Stochastic Modeling of Flow-Structure Interactions Using Generalized Polynomial Chaos

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We present a generalized polynomial chaos algorithm to model the input uncertainty and its propagation in flow-structure interactions. The stochastic input is represented spectrally by employing orthogonal polynomial functionals from the Askey scheme as the trial basis in the random space. A standard Galerkin projection is applied in the random dimension to obtain the equations in the weak form. The resulting system of deterministic equations is then solved with standard methods to obtain the solution for each random mode. This approach is a generalization of the original polynomial chaos expansion, which was first introduced by N. Wiener (1938) and employs the Hermite polynomials (a subset of the Askey scheme) as the basis in random space. The algorithm is first applied to second-order oscillators to demonstrate convergence, and subsequently is coupled to incompressible Navier-Stokes equations. Error bars are obtained, similar to laboratory experiments, for the pressure distribution on the surface of a cylinder subject to vortex-induced vibrations. [DOI: 10.1115/1.1436089]

1 Introduction

In the last decade there has been substantial progress in simulations of flow-structure interactions involving the full Navier-Stokes equations, e.g. [1,2]. While such simulations are useful in complementing experimental studies in the low Reynolds number range, they are based on *ideal* boundary conditions and *precisely* defined properties of the structure. In practice, such flow conditions and properties can only be defined approximately. As an example, the internal structural damping for the structure is typically taken as 1–3 percent of the critical damping since it cannot be quantified by direct measurements. It is, therefore, of great interest to formally model such uncertainty of stochastic inputs, and to formulate algorithms that reflect accurately the propagation of this uncertainty [3].

To this end, the Monte Carlo approach can be employed but it is computationally expensive and is only used as the last resort. The sensitivity method is a more economical approach, based on the moments of samples, but it is less robust and depends strongly on the modeling assumptions [4]. One popular technique is the perturbation method where all the stochastic quantities are expanded around their mean via Taylor series. This approach, however, is limited to small perturbations and does not readily provide information on high-order statistics of the response. The resulting system of equations becomes extremely complicated beyond second-order expansion. Another approach is based on expanding the inverse of the stochastic operator in a Neumann series, but this too is limited to small fluctuations, and even combinations with the Monte Carlo method seem to result in computationally prohibitive algorithms for complex systems [5].

A more effective approach pioneered by Ghanem and Spanos [6] in the context of finite elements for solid mechanics is based on a spectral representation of the uncertainty. This allows high-order representation, not just first-order as in most perturbation-based methods, at high computational efficiency. It is based on the original theory of Wiener (1938) on homogeneous chaos [7,8]. This approach was employed in turbulence in the 1960s [9–11]. However, it was realized that the chaos expansion converges slowly for turbulent fields [12–14], so the polynomial chaos approach did not receive much attention for a long time.

In more recent work [15,16] the polynomial chaos concept was extended to represent many different distribution functions. This generalized polynomial chaos approach, also referred as the Askey-chaos, employs the orthogonal polynomials from the Askey scheme [17] as the trial basis in the random space. The original polynomial chaos can be considered as a subset of the generalized polynomial chaos, as it employs Hermite polynomials, a subset of the Askey scheme, as the trial basis. In [15], the framework of Askey-chaos was proposed and convergence properties of different random bases were examined. In [16] the Askey-chaos was applied to model uncertainty in incompressible Navier-Stokes equations. Various tests were conducted to demonstrate the convergence of the chaos expansion in prototype flows.

For flow-structure interactions the interest on stochastic modeling so far has primarily been on the dynamics of lumped systems, i.e., single- or two-degree-of-freedom second-order oscillators [18,19]. The effect of the flow has been modeled via an interaction (source) term as either white noise or as a Gaussian distribution if the loading is caused by wind [20–23]. However, non-Gaussian distribution behavior for the response has been documented with *the excess index* well above or below zero (sharp or flat intermittency) [18]. For example, even for a velocity field following a Gaussian distribution, which is a reasonable assumption for maritime winds [21], the corresponding force given by the Morison formula

$$F_V(t)=\tfrac{1}{2}\rho D C_d V(t)|V(t)|$$

does not follow a Gaussian distribution. This is because the above formula defines a nonlinear (memoryless) transformation [20], and its first-density function is given by

$$f_1(v)=\frac{1}{2\sigma_V\sqrt{2\pi|v|}}\exp\left[-\frac{1}{2}\left(\frac{sign(v)\sqrt{|v|}-m_V}{\sigma_V}\right)^2\right],$$

where m_V and σ_V are the mean value and standard deviation of the the Gaussian distribution for the velocity $V(t)$.

In this paper we apply the chaos expansions to coupled Navier-Stokes/structure equations. We first demonstrate the convergence of chaos expansions by solving a second-order ordinary differential equation. We then present the stochastic modeling of the fully coupled flow-structure interaction problem for vortex-induced vi-

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brations in flow past a cylinder. The algorithms developed here are general and can be applied to any type of distributions although our applications are concentrated on Gaussian type random inputs.

In the next section we review the theory of the generalized polynomial chaos. In Section 3 we apply it to second-order oscillators, and in Section 4 we present its application to Navier-Stokes equations. In Section 5 we present the computational results of stochastic flow-structure interactions, and we conclude with a brief discussion in Section 6.

2 The Generalized Polynomial Chaos

In this section we introduce the generalized polynomial chaos expansion along with the Karhunen-Loeve (KL) expansion, another classical technique for representing random processes. The KL expansion can be used in some cases to represent efficiently the known stochastic fields, i.e., the stochastic inputs.

2.1 The Askey Scheme. The Askey scheme, which is represented as a tree structure in Fig. 1 (following [24]), classifies the hypergeometric orthogonal polynomials and indicates the limit relations between them. The "tree" starts with the Wilson polynomials and the Racah polynomials on the top. The Wilson polynomials are continuous while the Racah polynomials are discrete. The lines connecting different polynomials denote the limit transition relationships between them; this implies that the polynomials at the lower end of the lines can be obtained by taking the limit of one of the parameters from their counterparts on the upper end. For example, the limit relation between Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ and Hermite polynomials $H_n(x)$ is

$$\lim_{\alpha\to\infty} \alpha^{-1/2n} P_n^{(\alpha,\alpha)}\left(\frac{x}{\sqrt{\alpha}}\right) = \frac{H_n(x)}{2^n n!},$$

and between Meixner polynomials $M_n(x;\beta,c)$ and Charlier polynomials $C_n(x;a)$ is

$$\lim_{\beta\to\infty} M_n\left(x;\beta,\frac{a}{a+\beta}\right) = C_n(x;a).$$

For a detailed account of definitions and properties of hypergeometric polynomials, see [17]; for the limit relations of Askey scheme, see [25] and [24].

The orthogonal polynomials associated with the generalized polynomial chaos, include: Hermite, Laguerre, Jacobi, Charlier, Meixner, Krawtchouk, and Hahn polynomials.

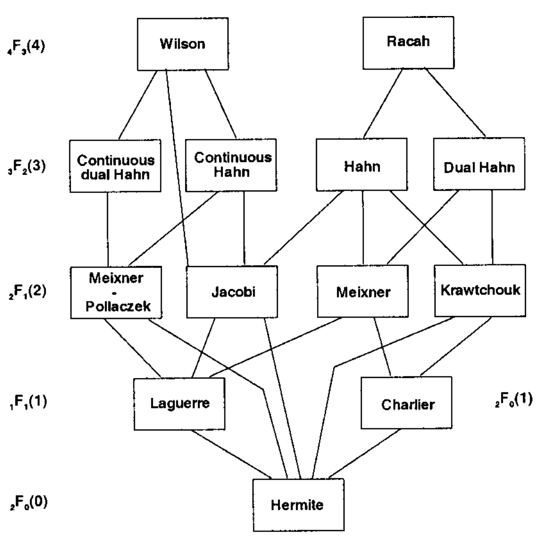

Fig. 1 The Askey scheme of orthogonal polynomials

Table 1 Correspondence of the type polynomials and random variables for different Askey-chaos ($N \geqslant 0$ is a finite integer).

	Random variables $\boldsymbol{\xi}$	Orthogonal polynomials $\{I_n\}$	Support
Continuous	Gaussian	Hermite	$(-\infty,\infty)$
	Gamma	Laguerre	$[0,\infty)$
	Beta	Jacobi	$[a,b]$
	Uniform	Legendre	$[a,b]$
Discrete	Poisson	Charlier	$\{0,1,2,\ldots\}$
	Binomial	Krawtchouk	$\{0,1,\ldots,N\}$
	Negative Binomial	Meixner	$\{0,1,2,\ldots\}$
	Hypergeometric	Hahn	$\{0,1,\ldots,N\}$

2.2 The Generalized Polynomial Chaos: Askey-Chaos. The original polynomial chaos [7,8] employs the Hermite polynomials in the random space as the trial basis to expand the stochastic processes. Cameron and Martin proved that such expansion converges to any second-order processes in the L_2 sense [26]. It can be seen from Fig. 1 that Hermite polynomial is a subset of the Askey scheme. The generalized polynomial chaos, or the Askey-Chaos, was proposed in [15,16] and employs more polynomials from the Askey scheme. Convergence to second-order stochastic processes can be readily obtained as a generalization of Cameron-Martin theorem [26].

A general second-order random process $X(\theta)$, viewed as a function of $\theta\in(0,1)$, i.e., the random event, can be represented in the form

$$X(\theta) = a_0 I_0 + \sum_{i_1=1}^{\infty} c_{i_1} I_1(\xi_{i_1}(\theta)) + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1} c_{i_1 i_2} I_2(\xi_{i_1}(\theta),\xi_{i_2}(\theta)) + \sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} c_{i_1 i_2 i_3} I_3(\xi_{i_1}(\theta),\xi_{i_2}(\theta),\xi_{i_3}(\theta)) + \ldots, \tag{1}$$

where $I_n(\xi_{i_1},\ldots,\xi_{i_n})$ denotes the Askey-chaos of order n in terms of the multi-dimensional random variables $\boldsymbol{\xi} = (\xi_{i_1},\ldots,\xi_{i_n})$. In the original polynomial chaos, $\{I_n\}$ are *Hermite* polynomials and $\boldsymbol{\xi}$ are *Gaussian* random variables. In the Askey-chaos expansion, the polynomials I_n are not restricted to Hermite polynomials and $\boldsymbol{\xi}$ not Gaussian variables. The corresponding type of polynomials and their associated random variables are listed in Table 1.

For notational convenience, we rewrite Eq. (1) as

$$X(\theta) = \sum_{j=0}^{\infty} \hat{c}_j \Phi_j(\boldsymbol{\xi}), \tag{2}$$

where there is a one-to-one correspondence between the functions $I_n(\xi_{i_1},\ldots,\xi_{i_n})$ and $\Phi_j(\boldsymbol{\xi})$, and their coefficients $\hat{c}_j$ and $c_{i_1,\ldots,i_r}$. Since each type of polynomials from the Askey scheme form a complete basis in the Hilbert space determined by their corresponding support, we can expect each type of Askey-chaos to converge to any L_2 functional in the L_2 sense in the corresponding Hilbert functional space as a generalized result of Cameron-Martin theorem ([26] and [27]). The orthogonality relation of the generalized polynomial chaos takes the form

$$\langle \Phi_i \Phi_j \rangle = \langle \Phi_i^2 \rangle \delta_{ij}, \tag{3}$$

where δ_{ij} is the Kronecker delta and $\langle\cdot,\cdot\rangle$ denotes the ensemble average which is the inner product in the Hilbert space of the random variables $\boldsymbol{\xi}$

$$\langle f(\boldsymbol{\xi}) g(\boldsymbol{\xi}) \rangle = \int f(\boldsymbol{\xi}) g(\boldsymbol{\xi}) W(\boldsymbol{\xi}) d\boldsymbol{\xi}, \tag{4}$$

or

$$\langle f(\boldsymbol{\xi})g(\boldsymbol{\xi})\rangle=\sum_{\xi} f(\boldsymbol{\xi})g(\boldsymbol{\xi})W(\boldsymbol{\xi}) \quad (5)$$

in the discrete case. Here $W(\boldsymbol{\xi})$ is the weighting function corresponding to the Askey polynomials chaos basis $\{\Phi_i\}$. Each type of orthogonal polynomials from the Askey-chaos has weighting functions of the same form as the probability function of its associated random variables $\boldsymbol{\xi}$, as shown in Table 1.

For example, as a subset of the Askey-chaos, the original polynomial chaos, also will be termed the Hermite-chaos, employs the Hermite polynomials defined as

$$I_n(\xi_{i_1},\ldots,\xi_{i_n})=e^{1/2\boldsymbol{\xi}^T\boldsymbol{\xi}}(-1)^n\frac{\partial^n}{\partial\xi_{i_1}\ldots\partial\xi_{i_n}}e^{-1/2\boldsymbol{\xi}^T\boldsymbol{\xi}}, \quad (6)$$

where $\boldsymbol{\xi}=(\xi_{i_1},\ldots,\xi_{i_n})$ are multi-dimensional independent Gaussian random variables with zero mean and unit variance. The weight function in the orthogonality relation (4) is

$$W(\boldsymbol{\xi})=\frac{1}{\sqrt{(2\pi)^n}}e^{-1/2\boldsymbol{\xi}^T\boldsymbol{\xi}}, \quad (7)$$

where n is the dimension of $\boldsymbol{\xi}$. It can seen that this is the same as the probability density function (PDF) of the n-dimensional Gaussian random variables. For example, the one-dimensional Hermite polynomials are:

$$\Psi_0=1, \quad \Psi_1=\xi, \quad \Psi_2=\xi^2-1, \quad \Psi_3=\xi^3-3\xi, \ \ldots \quad (8)$$

2.3 The Karhunen-Loeve Expansion. The Karhunen-Loeve (KL) expansion [28] is another way of representing a random process. It is a spectral expansion based on the decomposition of the covariance function of the process. Let us denote the process by $h(\mathbf{x},\theta)$ and its covariance function by $R_{hh}(\mathbf{x},\mathbf{y})$, where $\mathbf{x}$ and $\mathbf{y}$ are the spatial or temporal coordinates. By definition, the covariance function is real, symmetric, and positive definite. All eigenfunctions are mutually orthogonal and form a complete set spanning the function space to which $h(\mathbf{x},\theta)$ belongs. The KL expansion then takes the following form:

$$h(\mathbf{x},\theta)=\overline{h}(\mathbf{x})+\sum_{i=1}^{\infty}\sqrt{\lambda_i}\phi_i(\mathbf{x})\xi_i(\theta), \quad (9)$$

where $\overline{h}(\mathbf{x})$ denotes the mean of the random process, and $\xi_i(\theta)$ forms a set of independent random variables. Also, $\phi_i(\mathbf{x})$ and λ_i are the eigenfunctions and eigenvalues of the covariance function, respectively, i.e.,

$$\int R_{hh}(\mathbf{x},\mathbf{y})\phi_i(\mathbf{y})d\mathbf{y}=\lambda_i\phi_i(\mathbf{x}). \quad (10)$$

Among many possible decompositions of a random process, the KL expansion is optimal in the sense that the mean-square error of the finite term representation of the process is minimized. Its use, however, is limited as the covariance function of the solution process is often not known *a priori*. Nevertheless, the KL expansion provides an effective means of representing the input random processes when the covariance structure is known.

3 Second-order Random Oscillator

3.1 Governing Equations. We consider the second-order linear ordinary differential equation (ODE) system with both external and parametric random excitations.

$$\frac{dx}{dt}=y,$$

$$\frac{dy}{dt}+c(\theta)y+k(\theta)x=f(t,\theta), \quad (11)$$

where the parameters and forcing are functions of random event θ. We assume

$$c=\overline{c}+\sigma_c\xi_1, \quad k=\overline{k}+\sigma_k\xi_2,$$

$$f(t)=F\cos(\omega t)=(\overline{f}+\sigma_f\xi_3)\cos(\omega t), \quad (12)$$

where $(\overline{c},\sigma_c)$, $(\overline{k},\sigma_k)$ and $(\overline{f},\sigma_f)$ are the mean and standard deviation of c, k and F, respectively. The random variables ξ_1, ξ_2, and ξ_3 are assumed to be independent standard *Gaussian* random variables.

3.2 Chaos Expansions. By applying the generalized polynomial chaos expansion, we expand the solutions as

$$x(t)=\sum_{i=0}^{P}x_i(t)\Phi_i(\boldsymbol{\xi}), \quad y(t)=\sum_{i=0}^{P}y_i(t)\Phi_i(\boldsymbol{\xi}), \quad (13)$$

where we have replaced the infinite summation of $\boldsymbol{\xi}$ in infinite dimensions in Eq. (2) by a truncated finite-term summation of $\boldsymbol{\xi}$ in finite dimensional space. In this case, $\boldsymbol{\xi}=(\xi_1,\xi_2,\xi_3)$ is a three-dimensional *Gaussian* random vector according to the random inputs. This results in a three-dimensional *Hermite*-chaos expansion. The most important aspect of the above expansion is that the random processes have been decomposed into a set of deterministic functions in the spatial-temporal variables multiplied by the random basis polynomials which are independent of these variables:

$$\sum_{k=0}^{P}\frac{dx_k}{dt}\Phi_k=\sum_{k=0}^{P}y_k\Phi_k,$$

$$\sum_{k=0}^{P}\frac{dy_k}{dt}\Phi_k+\sum_{i=0}^{P}\sum_{j=0}^{P}c_iy_j\Phi_i\Phi_j$$

$$+\sum_{i=0}^{P}\sum_{j=0}^{P}k_ix_j\Phi_i\Phi_j=\sum_{k=0}^{P}f_k(t)\Phi_k, \quad (14)$$

where c_i, k_i, and f_i are the chaos expansion, similar to Eq. (13), of c, k, and f, respectively. A Galerkin projection of the above equation onto each polynomial basis $\{\Phi_i\}$ is then conducted in order to ensure the error is orthogonal to the functional space spanned by the finite-dimensional basis $\{\Phi_i\}$. By projecting with Φ_k for each $k=\{0,\ldots,P\}$ and employing the orthogonality relation (3), we obtain for each $k=0,\ldots,P$,

$$\frac{dx_k}{dt}=y_k,$$

$$\frac{dy_k}{dt}+\frac{1}{\langle\Phi_k^2\rangle}\sum_{i=0}^{P}\sum_{j=0}^{P}(c_iy_j+k_ix_j)e_{ijk}=f_k(t), \quad (15)$$

where $e_{ijk}=\langle\Phi_i\Phi_j\Phi_k\rangle$. Together with $\langle\Phi_i^2\rangle$, the coefficients e_{ijk} can be evaluated analytically from the definition of Φ_i. Equation (15) is a set of $(P+1)$ coupled ODEs. The total number of equation is determined by the dimensionality of the chaos expansion (n), in this case $(n=3)$, and the highest order (p) of the polynomials $\{\Phi\}$ [6]:

$$P=\sum_{s=1}^{p}\frac{1}{s!}\prod_{r=0}^{s-1}(n+r). \quad (16)$$

3.3 Numerical Results. The above set of equations can be integrated by any conventional method, e.g., Runge-Kutta. Here we employ the Newmark scheme which is second-order accurate in time. We define two error measures for the mean and variance of the solution