2 Polynomial Chaos Representation

In the proposed model, the identified random parameters of the system as well as the excitation and the response are treated as processes belonging to a finite energy Hilbert subspace of square-integrable random variables (Θ) that belongs to the general probability space of random variables $\{\Omega, \mathcal{F}, \mathcal{P}\}$. The subspace Θ represents the base to construct second-order random processes which, from a physical point of view are processes with finite "energy".

To understand these concepts, we start with some definitions:

- $\boldsymbol{\{\xi_i\}_{i=1}^{\infty}}$: Set of mutually independent Normal distributed random variables.
- $\widehat{\boldsymbol{\Psi}}_{\boldsymbol{p}}$: Space of all polynomials in $\{\xi_i\}$ of degree less or equal to p.
- $\widehat{\boldsymbol{\Psi}}_{\boldsymbol{p-1}}$: Space of all polynomials in $\{\xi_i\}$ of degree less or equal p-1.
- $\boldsymbol{\Psi_p}$: Polynomial Chaos of order p (Set of all polynomials in $\widehat{\boldsymbol{\Psi}}_{\boldsymbol{p}}$ orthogonal to $\widehat{\boldsymbol{\Psi}}_{\boldsymbol{p-1}}$).
- $\overline{\boldsymbol{\Psi}}_{\boldsymbol{p}}$: pth Homogeneous Chaos (Space spanned by Ψ_p).

Under general conditions, any element of Θ can be approximated arbitrarily closed (in the mean square sense) by elements of the ring of functions generated by the Hilbert Space spanned by the set of Normal Random Variables $\{\xi_i\}_{i=1}^{\infty}$ [2, 5]. This approximation is known as Polynomial Chaos (PC) [1, 2, 5, 3] and its main feature is its ability to capture in a few terms the basic random properties of the stochastic processes, situation that turns out to be very important when the system's randomness can be only characterized by expensive measurements.

Following this procedure a general second order random process $X(t, \theta)$, viewed as a function of θ (the random observation) and t, the index set, can be represented as

$$X(t,\theta) = \hat{x}_0(t)\Psi_0 + \underbrace{\sum_{i_1=1}^{\infty} \hat{x}_{i_1}(t)\Psi_1(\xi_{i_1})}_{1^{\text{st}}\text{order terms}} + \underbrace{\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1} \hat{x}_{i_1 i_2}(t)\Psi_2(\xi_{i_1}, \xi_{i_2})}_{2^{\text{nd}}\text{order terms}}$$
$$+ \underbrace{\sum_{i_1=1}^{\infty}\sum_{i_2=1}^{i_1}\sum_{i_3=1}^{i_2} \hat{x}_{i_1 i_2 i_3}(t)\Psi_3(\xi_{i_1}, \xi_{i_2}, \xi_{i_3})}_{3^{\text{rd}}\text{order terms}} \quad + \quad \dots \tag{1}$$

where $\Psi_p(\xi_{i_1}, \xi_{i_2}, \dots, \xi_{i_n})$ denotes a multidimensional Hermite polynomial of order p in terms of the vector $\boldsymbol{\xi}^T = \{\xi_{i_1}, \xi_{i_2}, \dots, \xi_{i_n}\}$, with each ξ_{i_k} representing a set of independent, identically distributed (iid) Gaussian random variables with zero mean and standard deviation. In this case the PC expansion is known as *Hermite Chaos* in reference to the orthogonal polynomials family used. This representation is basically a discretization of the original Wiener polynomial chaos

expansion [5].
As an example, a two dimensional expansion ($\boldsymbol{\xi} = \{\xi_1, \xi_2\}$) is shown, where the terms with fewer dimensions and fewer order are put at the beginning

$$
\begin{aligned}
X(t,\theta) = {} & \hat{x}_0(t)\,\Psi_0 + \hat{x}_1(t)\,\Psi_1(\xi_1) + \hat{x}_2(t)\,\Psi_1(\xi_2) \\
& + \hat{x}_{11}(t)\,\Psi_2(\xi_1,\xi_1) + \hat{x}_{12}(t)\,\Psi_2(\xi_1,\xi_2) + \hat{x}_{22}(t)\,\Psi_2(\xi_2,\xi_2) \\
& + \hat{x}_{111}(t)\,\Psi_3(\xi_1,\xi_1,\xi_1) + \hat{x}_{112}(t)\,\Psi_3(\xi_1,\xi_1,\xi_2) + \hat{x}_{122}(t)\,\Psi_3(\xi_1,\xi_2,\xi_2) + \hat{x}_{222}(t)\,\Psi_3(\xi_2,\xi_2,\xi_2) + \dots
\end{aligned}
$$

Note that $\hat{x}$'s sub indexes have a direct relation with their correspondent multidimensional Hermite polynomial $\Psi_p(\boldsymbol{\xi})$. For notational convenience, this last expression is recast to

$$
\begin{aligned}
X(t,\theta) = {} & x_0(t)\Psi_0 + x_1(t)\,\Psi_1(\boldsymbol{\xi}) + x_2(t)\,\Psi_1(\boldsymbol{\xi}) + x_3(t)\,\Psi_3(\boldsymbol{\xi}) + x_4(t)\,\Psi_4(\boldsymbol{\xi}) + x_5(t)\,\Psi_5(\boldsymbol{\xi}) \\
& + x_6(t)\,\Psi_6(\boldsymbol{\xi}) + x_7(t)\,\Psi_7(\boldsymbol{\xi}) + x_8(t)\,\Psi_8(\boldsymbol{\xi}) + x_9(t)\,\Psi_9(\boldsymbol{\xi}) + \dots
\end{aligned}
$$

or simply to

$$
\boxed{X(t,\theta) = \sum_{p=0}^{\infty} x_p(t)\Psi_p(\boldsymbol{\xi})} \tag{2}
$$

The sub indexes p in (2) represent the order of each multidimensional Hermite polynomial *only* when $X(t,\theta)$ is expanded in one dimension ($\boldsymbol{\xi} = \{\xi_1\}$). In the general multidimensional case ($\boldsymbol{\xi}^T = \{\xi_1, \xi_2, \dots, \xi_n\}$), the subscript p is only an index that indicates a place in the expansion.

Although the *Hermite Chaos* expansion works theoretically in any case [2], in general the rate of convergence can be significantly improved choosing a set of independent random variables with a probability distribution different than Gaussian, and changing the corresponding family of orthogonal polynomials (Laguerre, Legendre, Jacobi, etc.). As it is described in [3], the particular choice for the couple *Random Variable - Orthogonal Polynomial Family* has a direct relation with the type of uncertainties in the system and their probability distribution, helping significantly in some cases to improve the rate of convergence. Although this represents a very important topic in the computational simplification of the model, in this work the *Hermite-Chaos* polynomials will be used.

2.1 Coefficients PC expansion

Assume the following one-dimensional PC expansion is

$$
a = \sum_{k=0}^{P} A_i \Psi_k(\xi) \qquad A_k = \frac{\left\langle a, \Psi_k(\xi) \right\rangle}{\left\langle \Psi_k^2(\xi) \right\rangle} \tag{3}
$$

$\Psi_k(\xi)$ represents a Hermite polynomial in the standard Gaussian random variable ξ, i.e. $\{\Psi_0(\xi) = 0,\ \Psi_1(\xi) = \xi,\ \Psi_2(\xi) = \xi^2 - 1, \dots\}$. $\left\langle \Psi_k^2(\xi) \right\rangle$ only depends on k and can be obtained from tables or following the procedure explained in Appendix A for any ξ_i.

$\langle \cdot, \cdot \rangle$ denotes the inner product $E[a \cdot b]$ in the Hilbert space spanned by the chaos Hermite basis $\{\Psi_k(\xi_i)\}$, i.e.

$$\langle a(\xi),\, b(\xi) \rangle = \int_\Omega a(\xi) \cdot b(\xi)\, dF_\xi \qquad \xi \in \Omega$$

then from (3)

$$A_{ik} = \frac{\langle a_i,\, \Psi_k(\xi_i) \rangle}{\langle \Psi_k^2(\xi_i) \rangle} = \frac{1}{\langle \Psi_k^2 \xi_i) \rangle} \int_\Omega a_i\, \Psi_k(\xi_i)\, dF_{\xi_i}$$

To calculate this integral using Gauss quadrature, the cumulative distribution function (*cdf*) of every a_i ($C_i(a_i)$) is numerically calculated using the N_{OBS} previous observations. Then, an approximation to $C_i^{-1}(\theta)$ ($\theta \in [0,\, 1]$) is found using piecewise polynomials on each a_i's histogram. Finally, $C_i^{-1}(\theta)$ and ξ_i's inverse *cdf* ($E^{-1}(\theta)$) are evaluated on the Gauss points of the quadrature to obtain an approximation to each A_{ik}

$$A_{ik} = \frac{1}{\langle \Psi_k^2(\xi) \rangle} \int_0^1 C_i^{-1}(\theta)\, \Psi_k(E^{-1}(\theta))\, d\theta$$

Numerical Results: The proposed approach was numerically implemented with the same initial conditions as in the first example; $c \sim U(\mu_c,\, \sigma_c)$ (Uniformly distributed). In this case the value of the standard deviations will be consider equal to the 50 percent of their mean value, so $\mu_c = 1.2457 \Rightarrow \sigma_c = 0.6$ ($\Rightarrow\ a = 0.1669,\, b = 2.3245$) . To find the PC coefficients of the uniformly distributed dumping ratio c (c_j), we project c onto the random space spanned by $\Psi_i(\xi_1)$:

$$c = \sum_{i=0}^{P} c_j \Psi_j(\xi_1)$$

$$\therefore \quad c_j = \frac{\left\langle c,\, \Psi_j(\xi_1) \right\rangle}{\left\langle \Psi_j^2(\xi_1) \right\rangle} = \frac{\left\langle c,\, \Psi_j(\xi_1) \right\rangle}{j!} = \frac{1}{j!} \int_0^1 C^{-1}(u) \Psi_j(G^{-1}(u))\, du$$

where $C^{-1}(u)$ represents the inverse function of c's cumulative distribution function (*cdf*), and $G^{-1}(u)$ the inverse *cdf* of ξ_1's, that is just the inverse of a Gaussian standard random variable. This integration was performed using Gauss quadrature and the following values were found:

$$c_0 = 1.2457$$
$$c_1 = 0.6086478$$
$$c_2 = 0$$
$$c_3 = -0.0507207$$

The remaining c_j coefficients are negligible, so they are set equal to 0.

A Hermite Polynomials

In the course of the current research, several formulas have been used and derived involving orthogonal properties of Hermite polynomials, simplifying considerably the computational effort. Here a summary of those formulas is presented.

A.1 One Dimensional Hermite Polynomials

1. **Explicit formula:** One dimensional Hermite polynomial of order n [13]:

$$\Psi_n(\xi) = \sum_{j=0}^{[n/2]} (-1)^j \, \frac{n!}{2^j \, j! \, (n-2j)!} \cdot \xi^{n-2j} \qquad n = 0,\, 1,\, \ldots \tag{4}$$

where $[n/2] = \begin{cases} n/2 & n \text{ even} \\ (n-1)/2 & n \text{ odd} \end{cases}$

Therefore:
$$\begin{aligned} \Psi_0(\xi) &= 1 \\ \Psi_1(\xi) &= \xi \\ \Psi_2(\xi) &= \xi^2 - 1 \\ \Psi_3(\xi) &= \xi^3 - 3\xi \\ \Psi_4(\xi) &= \xi^4 - 6\xi^2 + 3 \\ &\ldots \end{aligned}$$

2. **Orthogonality:** The orthogonality relation takes the form

$$\Big\langle \Psi_i(\xi), \Psi_j(\xi) \Big\rangle = \delta_{ij} \Big\langle \Psi_i^2(\xi) \Big\rangle = \delta_{ij} \, i! \tag{5}$$

where δ_{ij} is the Kronecker delta and $\Big\langle \cdot \,, \cdot \Big\rangle$ denotes the ensamble average, which is the inner product in the Hilbert space of the standard gaussian random variable ξ,

$$\Big\langle f(\xi),\, g(\xi) \Big\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\xi^2/2} \, f(\xi) \, g(\xi) d\xi \tag{6}$$

3. **Triple Inner Product:** [16]

$$\Big\langle \Psi_\alpha(\xi), \Psi_\beta(\xi), \Psi_\gamma(\xi) \Big\rangle = \begin{cases} \dfrac{\alpha! \, \beta! \, \gamma!}{(s-\alpha)!(s-\beta)!(s-\gamma)!} & \begin{cases} \alpha + \beta + \gamma \quad \text{is even} \\ s \triangleq \dfrac{\alpha+\beta+\gamma}{2} \\ s \geq \alpha, \quad s \geq \beta, \quad s \geq \gamma \end{cases} \\ 0 & \text{Otherwise} \end{cases} \tag{7}$$

A.2 Multidimensional Case

1. **Explicit Formula:** The general expression for a multidimensional Hermite polynomial of dimension n ($\boldsymbol{\xi}^T = \{\xi_{i_1}, \ldots, \xi_{i_n}\}$), order q, that contains only p variables of the vector $\boldsymbol{\xi}$ ($1 \leq p \leq n$), is given by the Rodriguez formula:

$$\Psi(\boldsymbol{\xi}) = \Psi(\xi_{i_1}, \xi_{i_2}, \ldots, \xi_{i_n}) = (-1)^q \, e^{(1/2)\boldsymbol{\xi}^T \boldsymbol{\xi}} \, \frac{\partial^q}{\partial \xi_{i_1} \cdots \partial \xi_{i_p}} \, e^{-(1/2)\boldsymbol{\xi}^T \boldsymbol{\xi}} \tag{8}$$

After examine the way that every term is derived in equation (8), it can be proved that any multidimensional Hermite polynomial of dimension n and order q is equal to the multiplication of n one-dimensional Hermite polynomials in terms of the set of variables $\{\xi_{i_1}, \ldots, \xi_{i_n}\}$

$$\Psi(\boldsymbol{\xi}) = \Psi_{q_1}(\xi_{i_1}) \cdots \Psi_{q_j}(\xi_{i_j}) \cdots \Psi_{q_n}(\xi_{i_n}) \qquad \text{where} \begin{cases} q_1 + \ldots + q_j + \ldots + q_n = q \\ q_j \,\epsilon\, \{0,\, 1,\, \ldots,\, q\} \\ \xi_{i_j} \,\epsilon\, \{\xi_{i_1},\, \xi_{i_2},\, \ldots,\, \xi_{i_n}\} \end{cases} \tag{9}$$

In general, there are $C = \binom{q+n-1}{q}$ possible combinations of the variables in $\{\xi_{i_1}, \ldots, \xi_{i_n}\}$ such that the sum $q_1 + \ldots + q_j + \ldots + q_n = q$, therefore for every new order q that we want to add to our expansion in (2) (to improve the accuracy of the results), we should add C new terms. If we want to calculate the Hermite polynomial of dimension $n = 4$ ($\boldsymbol{\xi}^T = \{\xi_1, \xi_2, \xi_3, \xi_4\}$) and order $q = 3$, we see that $C = 20$ possible combinations of the 4 variables in $\boldsymbol{\xi}$ that create different polynomials with order 3. Here we show 2 of those polynomials, the combination $\{\xi_1, \xi_1, \xi_1\}$

$$\Psi(\xi_1, \xi_2, \xi_3, \xi_4) = (-1)^3 \, e^{(1/2)\boldsymbol{\xi}^T \boldsymbol{\xi}} \, \frac{\partial^3}{\partial \xi_1^3} \, e^{-(1/2)\boldsymbol{\xi}^T \boldsymbol{\xi}} = \xi_1^3 - 3\xi_1 \tag{10}$$

and the combination $\{\xi_1, \xi_2, \xi_4\}$

$$\Psi(\xi_1, \xi_2, \xi_3, \xi_4) = (-1)^3 \, e^{(1/2)\boldsymbol{\xi}^T \boldsymbol{\xi}} \, \frac{\partial^3}{\partial \xi_1 \, \partial \xi_2 \, \partial \xi_4} \, e^{-(1/2)\boldsymbol{\xi}^T \boldsymbol{\xi}} = \xi_1 \xi_2 \xi_4 \tag{11}$$

The total number of polynomials of dimension n and highest order q is $\frac{(n+q)!}{n!\,q!}$.

<u>**Example:**</u> In this example we show all the multidimensional Hermite polynomials of dimension n=2 of order at most equal to 3 (q=3). Based on the equation (9), every polynomial of order q will have the form

$$\Psi(\xi_1, \xi_2) = \Psi_a(\xi_1) \, \Psi_b(\xi_2) \qquad \text{where} \begin{cases} q = \{0, 1, 2, 3\} \\ a, b \,\epsilon\, \{0, 1, \ldots, q\} \\ \text{and } a + b = q \end{cases} \tag{12}$$

Index $\boldsymbol{i}$	Order $\boldsymbol{q}$	# of terms $\boldsymbol{C}$	a	b	$\boldsymbol{\Psi_a}(\xi_1)\boldsymbol{\Psi_b}(\xi_2)$	$\boldsymbol{\Psi_i}(\boldsymbol{\xi})$
0	0	1	0	0	$\Psi_0(\xi_1)\Psi_0(\xi_2)$	1
1	1	2	1	0	$\Psi_1(\xi_1)\Psi_0(\xi_2)$	ξ_1
2			0	1	$\Psi_0(\xi_1)\Psi_1(\xi_2)$	ξ_2
3	2	3	2	0	$\Psi_2(\xi_1)\Psi_0(\xi_2)$	$\xi_1^2 - 1$
4			1	1	$\Psi_1(\xi_1)\Psi_1(\xi_2)$	$\xi_1\,\xi_2$
5			0	2	$\Psi_0(\xi_1)\Psi_2(\xi_2)$	$\xi_2^2 - 1$
6	3	4	3	0	$\Psi_3(\xi_1)\Psi_0(\xi_2)$	$\xi_1^3 - 3\xi_1$
7			2	1	$\Psi_2(\xi_1)\Psi_1(\xi_2)$	$(\xi_1^2 - 1)\xi_2$
8			1	2	$\Psi_1(\xi_1)\Psi_2(\xi_2)$	$\xi_1(\xi_2^2 - 1)$
9			0	3	$\Psi_0(\xi_1)\Psi_3(\xi_2)$	$\xi_2^3 - 3\xi_2$

Table 1: *Multidimensional Hermite polynomials of dimension 2, highest order* $q = 3$

Table 1 shows all these polynomials. The index $\boldsymbol{i}$ represents the same as in equation (2), just a position in the expansion, not the order of the polynomial.

This way of representing multidimensional Hermite polynomials will prove to be very useful in future.

2. **Orthogonality:** Based on the definition of inner product (6) and taking

$$\Psi_p(\boldsymbol{\xi}) = \Psi_{s_1}(\xi_{i_1})\,\cdots\,\Psi_{s_j}(\xi_{i_j})\,\cdots\,\Psi_{s_n}(\xi_{i_n}) \qquad \text{where} \begin{cases} s_1 + \ldots + s_j + \ldots + s_n = s \\ s_j \,\epsilon \,\{0,\,1,\,\ldots,\,s\} \\ \xi_{i_j} \,\epsilon\, \{\xi_{i_1},\,\xi_{i_2},\,\ldots,\,\xi_{i_n}\} \end{cases}$$

$$\Psi_m(\boldsymbol{\xi}) = \Psi_{q_1}(\xi_{i_1})\,\cdots\,\Psi_{q_j}(\xi_{i_j})\,\cdots\,\Psi_{q_n}(\xi_{i_n}) \qquad \text{where} \begin{cases} q_1 + \ldots + q_j + \ldots + q_n = q \\ q_j \,\epsilon \,\{0,\,1,\,\ldots,\,q\} \\ \xi_{i_j} \,\epsilon\, \{\xi_{i_1},\,\xi_{i_2},\,\ldots,\,\xi_{i_n}\} \end{cases}$$

we have

$$\begin{aligned} \Big\langle \Psi_p(\boldsymbol{\xi}), \Psi_m(\boldsymbol{\xi}) \Big\rangle &= \Big\langle \Psi_{q_1}(\xi_{i_1})\,\cdots\,\Psi_{q_j}(\xi_{i_j})\,\cdots\,\Psi_{q_n}(\xi_{i_n}), \Psi_{s_1}(\xi_{i_1})\,\cdots\,\Psi_{s_j}(\xi_{i_j})\,\cdots\,\Psi_{s_n}(\xi_{i_n}) \Big\rangle \\ &= \Big\langle \Psi_{q_1}(\xi_{i_1}), \Psi_{s_1}(\xi_{i_1}) \Big\rangle \Big\langle \Psi_{q_2}(\xi_{i_2}), \Psi_{s_2}(\xi_{i_2}) \Big\rangle \cdots \Big\langle \Psi_{q_n}(\xi_{i_n}), \Psi_{s_n}(\xi_{i_n}) \Big\rangle \\ &= (\delta_{q_1 s_1} s_1!)\,(\delta_{q_2 s_2} s_2!)\,\cdots\,(\delta_{q_n s_n} s_n!) \end{aligned}$$

This last product is different than 0 only if $q_j = s_j,\ \forall\, j\,\epsilon\,\{1, 2, \ldots, n\}$, therefore

$$\Big\langle \Psi_p(\boldsymbol{\xi}), \Psi_m(\boldsymbol{\xi}) \Big\rangle = \begin{cases} s_1! s_2! \cdots s_n! & \text{iff} \quad q_j = s_j\; \forall\, j\,\epsilon\,\{1, 2, \ldots, n\} \\ 0 & \text{Otherwise} \end{cases} \tag{13}$$

3. **Triple Inner Product:** From the previous equations is easy to show that

$$\Big\langle \Psi_q(\boldsymbol{\xi}), \Psi_s(\boldsymbol{\xi}), \Psi_t(\boldsymbol{\xi}) \Big\rangle = \Big\langle \Psi_{q_1}(\xi_{i_1}), \Psi_{s_1}(\xi_{i_1}), \Psi_{t_1}(\xi_{i_1}) \Big\rangle \cdots \Big\langle \Psi_{q_n}(\xi_{i_n}), \Psi_{s_n}(\xi_{i_n}) \Psi_{t_n}(\xi_{i_n}) \Big\rangle$$

and each one-dimensional triple inner product on the right hand side can be evaluated using equation (7).

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