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Existence and Controllability For Fractional Systems

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Congratulations on Ivan's 60th Birthday !

Major Cooperation Research Fields with Ivan

- *Joint Research Field 1:*

  Limit Cycles

- *Joint Research Field 2:*

  Hemivariational Inequalities

- *Joint Research Field 3:*

  Fractional Calculus

Joint Publications on Limit Cycles

- LIU ZH, SAEZ E, SZANTO I., *A system of degree four with an invariant trangle and at least three small amplitude limit cycles, Electronic Journal of Qualitative Theory of Differential Equations, 2010, No. 69, 1-7.*
- LIU ZH, SAEZ E.,SZANTO I., *Limit cycles and invariant parabola in a kukles system of degree three, Acta Mathematica Scientia, 28B (3)(2008) 312–321.*
- LIU ZH, SAEZ E.,SZANTO I., *A cubic system with an invariant triangle surrounding at least one limit cycle, Taiwanese Journal of Mathematics, 7(2), 2003, 275-281.*

Joint Publications on Hemivariational Inequalities

- LIU ZH, SZANTO I., *Inverse coefficient problems for parabolic hemivariational inequalities, Acta Mathematica Scientia, 2011,31B(4):1318 − 1326.*
- LIU ZH, SZANTO I., *Multivalued differential equations in Banach spaces and their applications, Acta Mathematica Scientia, 22(2), 2002, 213-221.*

Joint Publications on Fractional Calculus

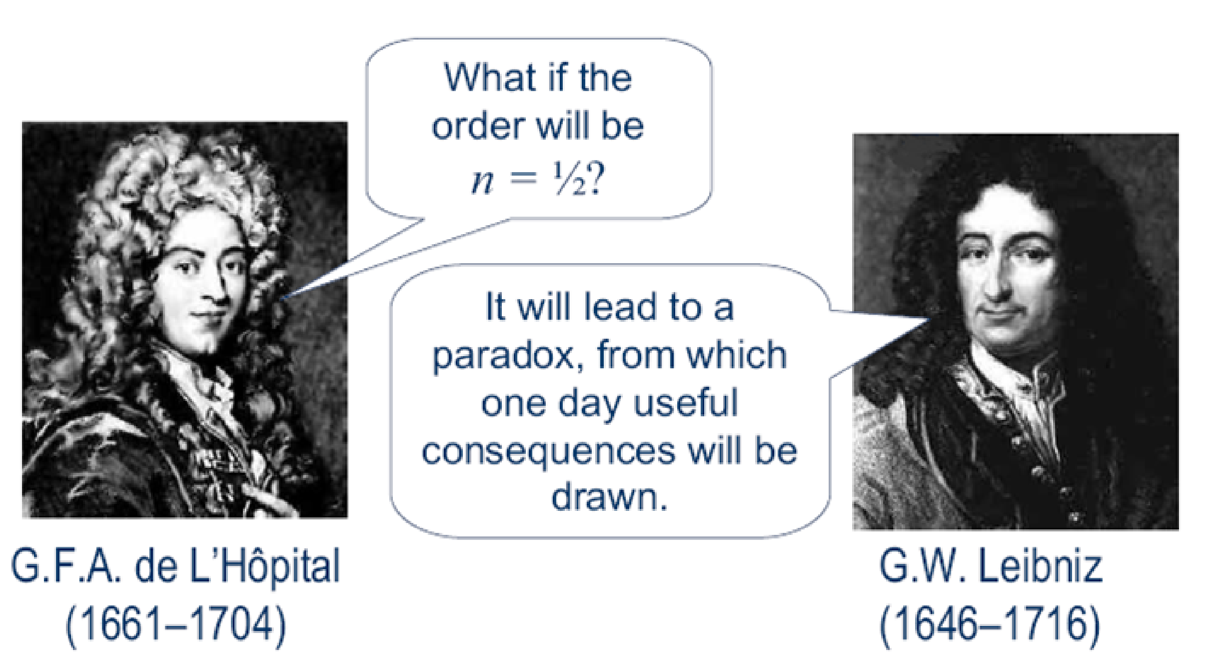
- LIU ZH, SZANTO I., *Existence of solutions for fractional impulsive differential equations with p-Laplacian operator, Acta Mathematica Hungarica, 2013, 1-17.*
- LIU ZH, SZANTO I., *Monotone Iterative Technique for Riemann – Liouville Fractional Integro-Differential Equations with Advanced Arguments, Results. Math. 63 (2013), 1277 – 1287.*

Existence and Controllability For Fractional Systems

1 Introduction

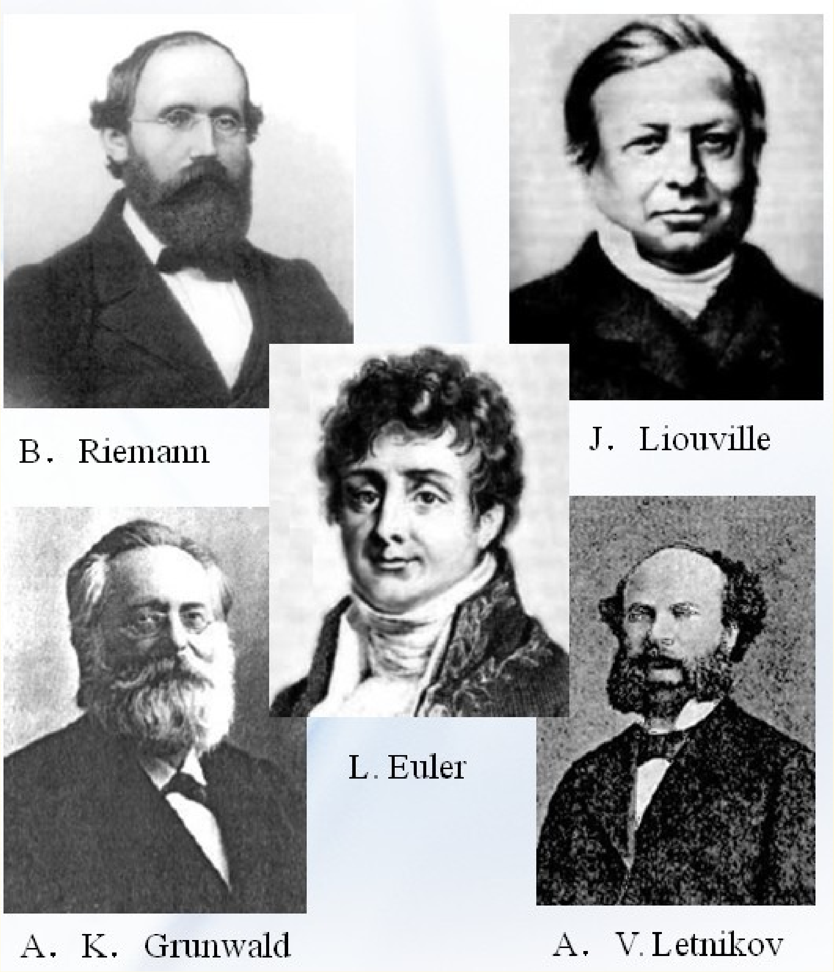
- *What is fractional calculus?*

In a letter to Leibniz in 1695 L'Hopital raised the following question: "Can the meaning of derivatives with integer order be generalized to derivatives with non-integer orders?"



Many famous mathematicians made great contributions to fractional calculus

- L. Euler,
- P.S. Laplace,
- J.B.J. Fourier,
- N.H. Abel,
- J. Liouville,
- B. Riemann,
- A. K. Grunwald,
- A. V. Letnikov,
- H. Weyl,
- . . .



Fractional Integral

- *Riemann (1847):*

$$I_{0^+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad \alpha > 0, \quad (1.1)$$

is called Riemann-Liouville fractional integral of order α*, where* Γ *is the gamma function.*

- *If* α *is an integer, then (1.1) is the general integer-order integral with the order* α*.*

Definitions of Fractional Derivatives

- *Riemann-Liouville Fractional Derivatives:*

For a function $f(t)$ *given in the interval* $[0,\infty)$, *the expression*

$$ {}^{L}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} f(s)dt, \tag{1.2}$$

where $n=[\alpha]+1$, $[\alpha]$ *denotes the integer part of number* α, *is called the Riemann-Liouville fractional derivative of order* α.

- *Caputo Fractional Derivatives:*

For the function $f(t) \in C^n[0,\infty), f:[0,\infty) \to R$ *as*

$$^cD^\alpha f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds, \qquad (1.3)$$

where $t > 0, n = [\alpha]+1, [\alpha]$ *denotes the integer part of real number* α.

The Relationship between the two definitions

$$^cD^\alpha f(t) = {}^L\boldsymbol{D}_t^\alpha \left(f(t) - \sum_{k=0}^{n-1} \frac{t^k}{k!} f^{(k)}(0) \right).$$

2 Existence for Fractional Differential Systems

2.1 Existence for Caputo Fractional Evolution System

$$\begin{cases} {}^cD_t^\alpha x(t) = Ax(t) + f(t, x(t)), t \in J = [0, b], \\ x(0) = x_0, \quad 0 < \alpha \le 1 \end{cases} \tag{2.1}$$

- ${}^cD_t^\alpha$ *is the Caputo fractional derivative of order* α *with the lower limit zero.*
- A *is the infinitesimal generator of an analytic semigroup* $\{T(t), t \ge 0\}$ *on a Banach space* X.

Assumptions

- $H(1)$: $T(t)$ *is a compact operator for every* $t > 0$,
- $H(2)$: *There exist a function* $\phi(\cdot) \in L^p(J, R^+)$, $p > \frac{1}{\alpha}$ *and a constants* $c > 0$, *such that*

$$\|f(t,x)\| \leq \phi(t) + c\|x\|_X, \quad \textit{for a.e. } t \in J, \textit{ and all } x \in X.$$

- $H(3)$: *There exists a constant* $L > 0$ *such that*

$$\|f(t,x) - f(t,y)\| \leq L\|x - y\|_X, \quad \textit{for all } x,\ y \in X.$$

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[Liu ZH, Lv JY, Comp. Math. Appl., 62 (2012) 1063-1077]

Theorem 2.1 *Assume that the hypotheses* $H(1) - H(3)$ *are satisfied. Then the system (2.1) exits a mild solution on J.*

The ideas of the proof. *If $\alpha = 1$, the mild solution can be written as:*

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s, x(s)ds,$$

where $T(t) = e^{At}$.

By an iterative method, we show the mild solution of system (2.1) has the following form:

$$x(t) = S_\alpha(t)x_0 + \int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s, x(s))ds$$

where

$$S_\alpha(t) = \int_0^\infty \xi_\alpha(\theta) T(t^\alpha \theta) d\theta, \; T_\alpha(t) = \alpha \int_0^\infty \theta \xi_\alpha(\theta) T(t^\alpha \theta) d\theta,$$

and

$$\xi_\alpha(\theta) = \frac{1}{\alpha} \theta^{-(1+\frac{1}{\alpha})} \overline{\omega}_\alpha(\theta^{-\frac{1}{\alpha}}) \geq 0,$$

$$\overline{\omega}_\alpha(\theta) = \frac{1}{\pi} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-n\alpha-1} \frac{\Gamma(n\alpha+1)}{n!} \sin(n\pi\alpha), \; \theta \in (0,\infty),$$

ξ_α *is a probability density function defined on* $(0,\infty)$, *that is*

$$\xi_\alpha(\theta) \geq 0, \; \theta \in (0,\infty) \textit{ and } \int_0^\infty \xi_\alpha(\theta) d\theta = 1.$$

2.2 Existence for Riemann-Liouville Fractional System

$$\begin{cases} {}^L D_t^q x(t) = Ax(t) + f(t, x(t))), \;\; t \in J, \; 0 < q \leq 1, \\ I_{0^+}^{1-q} x(t)|_{t=0} = x_0 \in X, \end{cases} \tag{2.2}$$

- ${}^L D_t^q$ is the Riemann-Liouville fractional derivative of order q.
- $I_{0^+}^{1-q} x(t)|_{t=0} = \lim_{t\to 0} \frac{1}{\Gamma(1-q)} \int_0^t (t-s)^{-q} x(s) ds$
- This kind of initial conditions has a definite physical meaning and memory effects.

Assumptions

- $H(1)$: *$T(t)$ is a C_0-semigroup and $T(t)$ is continuous in the uniform operator topology for $t > 0$.*
- $H(2)$: *There exist a function $\phi(\cdot) \in L^p(J, R^+)$, $p > \frac{1}{\alpha}$ and a constants $c > 0$, such that*

$$\|f(t,x)\| \leq \phi(t) + ct^{1-\alpha}\|x\|_X, \text{ for a.e. } t \in J, \text{ and all } x \in X.$$

- $H(3)$: *There exists a constant $L > 0$ such that*

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\|_X, \quad \text{for all } x,\ y \in X.$$

[2013 Liu ZH, Commun. Nonlinear Sci. Numer. Simulat, 18 1362-1373]

[2013 Liu ZH, Sun JH, Szanto I.,Results. Math. 63, 1277－1287]

Theorem 2.2 *Assume that the hypotheses $H(1) - H(3)$ are satisfied. Then the system (2.2) exits a mild solution on J.*

The ideas of proof

Key: Define a new Banach space as:

$$C_{1-\alpha}(J,X)=\{x:t^{1-\alpha}x(t)\in C(J,X)\}$$

with the norm $\|x\|_{C_{1-\alpha}}=\sup\{t^{1-\alpha}\|x(t)\|_X : t\in J\}$.

The solution of (2.2) can be written as

$$x(t)=t^{\alpha-1}T_\alpha(t)x_0+\int_0^t (t-s)^{\alpha-1}T_\alpha(t-s)f(s,x(s))ds,\ t\in J.$$

where

$$T_\alpha(t) = \alpha \int_0^\infty \theta \xi_\alpha(\theta) T(t^\alpha \theta) d\theta,$$

and

$$\xi_\alpha(\theta) = \frac{1}{\alpha}\theta^{-(1+\frac{1}{\alpha})}\overline{\omega}_\alpha(\theta^{-\frac{1}{\alpha}}) \geq 0,$$

$$\overline{\omega}_\alpha(\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}\theta^{-n\alpha-1}\frac{\Gamma(n\alpha+1)}{n!}\sin(n\pi\alpha),\ \theta \in (0,\infty),$$

ξ_α *is a probability density function defined on* $(0,\infty)$, *that is*

$$\xi_\alpha(\theta) \geq 0,\ \ \theta \in (0,\infty) \textit{ and } \int_0^\infty \xi_\alpha(\theta) d\theta = 1.$$

3 Complete Controllability

The concept of controllability, when it was first introduced by Kalman in 1960, has been widely studied by many authors. Generally speaking, the significant meaning of the controllability lies on the fact that it can steer a dynamic control system from an arbitrary initial state to arbitrary final state using the set of admissible controls, and it plays an important role in deterministic and stochastic control theory and engineering.

Kalman

Consider the following fractional differential evolution inclusion systems:

$$\begin{cases} {}^cD_t^\alpha x(t) \in Ax(t) + Bu(t) + F(t, x(t)), \\ x(0) = x_0, t \in J = [0, b]. \end{cases} \tag{3.1}$$

- *The state* $x(\cdot)$ *takes values in* X*; the control function* $u(\cdot)$ *takes its value in* $L^{\frac{1}{p}}(J, U)(p > \frac{1}{\alpha})$ *of admissible control functions for a Banach space* $U, B : U \to X$ *is a bounded linear operator.*
- $F : J \times X \to \mathcal{P}_{cv,cp}(X)$.

- **Definition 3.1** *(Complete controllability) The fractional system (3.1) is said to be completely controllable on the interval* J*, iff for every* $x_0, x_1 \in X$*, there exists a control* $u \in L^{\frac{1}{p}}(J, U)$ *such that a mild solution* $x(t)$ *to system (3.1) satisfies* $x(b) = x_1$.

Assumptions:

- $H(1)$ $T(t)(t>0)$ *is a strongly continuous semigroup on* X.

- $H(2)$ *The linear operator* $W: L^{\frac{1}{p}}(J,U) \to X$, *defined by*

$$Wu := \int_0^b (b-s)^{\alpha-1} T_\alpha(b-s) Bu(s)ds,$$

has an inverse operator W^{-1} *which takes value in* $L^{\frac{1}{p}}(J,U)/kerW$ *and there exist two positive constants* $M_2, M_3 > 0$ *such that* $\|B\| \le M_2$, *and* $\|W^{-1}\| \le M_3$.

- $H(3)$ *For all bounded subsets* B_r, *the set*

$$\Pi^{\varepsilon,\delta}(t) = \Big\{ S_\alpha(t)(x_0 - h(0,x_0)) + \alpha \int_0^{t-\varepsilon} \int_\delta^\infty \theta(t-s)^{\alpha-1} \times \xi_\alpha(\theta) T((t-s)^\alpha \theta) f(s) d\theta ds : x \in B_r \Big\}$$

is relatively compact on $C(J,X)$ *for arbitrary* $\varepsilon > 0, t \in (0,b]$ *and any* $\delta > 0$.

[2013 Liu ZH, Li XW, J. Optim. Theory Appl. 156(2013) 167-182.]

Theorem 3.1 *Assume that the hypotheses* $H(1) - H(3)$ *are satisfied. Then the system (3.1) is completely controllable on* J.

Proof. *If* $x(0) = x_0$ *and there exists* $f \in F(t, x(t))$ *a.e. on* J, *then the mild solution of system (3.1) can be written as*

$$\begin{aligned} x(t) &= S_\alpha(t)(x_0 - h(0, x_0)) + h(t, x(t)) \\ &\quad + \int_0^t (t-s)^{\alpha-1} A T_\alpha(t-s) h(s, x(s)) ds \\ &\quad + \int_0^t (t-s)^{\alpha-1} T_\alpha(t-s) \Big[f(s) + Bu(s) \Big] ds \end{aligned} \tag{3.2}$$

The key step is to define the control u_x *as*

$$u_x(t) = W^{-1}\Big\{x_1 - S_\alpha(b)(x_0 - h(0, x_0)) - \int_0^b (b-s)^{\alpha-1} T_\alpha(b-s) f(s) ds - \int_0^b (b-s)^{\alpha-1} A T_\alpha(b-s) h(s, x(s)) ds\Big\}(t), \; t \in J$$

4 Approximate Controllability

For Caputo fractional differential evolution inclutions

$$\begin{cases} {}^cD^q x(t) \in Ax(t) + Bu(t) + F(t, x(t)), \ \frac{1}{2} < q \leq 1, \ t \in J = [0, b], \\ x(0) = x_0. \end{cases} \tag{4.1}$$

- *The state* $x(\cdot)$ *takes values in* X, *the control function* $u(\cdot)$ *is given in* $L^2(J, U)$, *admissible control functions with* U *a real Hilbert space,* $B : U \to X$ *is a bounded linear operator.*
- $F : J \times X \to \mathcal{P}(X) := 2^X \setminus \{\emptyset\}$ *is a multivalued map.*

Recall the definitions:

$\mathscr{R}(b, x_0) = \{x(b; u); u \in L^2(J, U), x(0; u) = x_0\}$

is called the reachable set of system (4.1) *with the initial value* x_0 *at terminal time* b.

- *The system* (4.1) *is said to be approximate controllability on the interval* J *if for all* $x_0 \in X$ *we have* $\overline{\mathscr{R}(b, x_0)} = X$.
- *The system* (4.1) *is said to be complete controllable on* J, *if for all* $x_0 \in X$ *we have* $\mathscr{R}(b, x_0) = X$,

  i.e., there exists $u \in L^2(J, U)$ *such that the mild solution of* (4.1) *satisfies* $x(0; u) = x_0, \ x(b; u) = x_1$.

Hypotheses

(H_1): $T(t)$ *is a compact* C_0*-semigroup.*

(H_2): F *is a multivalued map satisfying* $F : J \times X \to \mathcal{P}_{cp,cv}(X)$ *is measurable to* t *for each fixed* $x \in X$*, u.s.c. to* x *for a.e.* $t \in J$*, and for each* $x \in C(J,X)$ *the set*

$$S_{F,x} = \{f \in L^1(J, X) : f(t) \in F(t, x)\}$$

is nonempty.

(H_3): *There exists a positive constant* L *and a bounded nonnegative measurable function* ϕ *such that* $\|F(t, x)\| \le \phi(t) + L\|x\|$ *for all* $(t,\ x) \in J \times X$.

Result

[2013, Liu ZH, Lv JY, R. Sakthivel], IMA J. Math. Control Inf. (2013)]

Theorem 4.1 *Assume that assumptions $(H_1) - (H_3)$ are satisfied, and the linear system corresponding to (4.1) is approximately controllable on J. Then system (4.1) is approximately controllable on J.*

5 Optimal feedback control problems

We consider the following problem:

$$\inf_{(x,u)} \varphi(x,u) = \left\{ \int_0^T L(t,x(t),u(t))dt \right\}$$

subject to

$$\begin{cases} {}^L D_t^q x(t) = Ax(t) + f(t,x(t),u(t)), \ \ t \in [0,T], \ 0 < q \le 1, \\ u(t) \in U(t, t^{1-q}x(t)), \ \ \text{a.e. } t \in [0,T], \\ I_{0^+}^{1-q} x(t)|_{t=0} = x_0 \in X, \end{cases}$$

- $f : [0, T] \times X \times V \to X$ *is given function to be specified later.*
- *The control function u is given in a suitable admissible control set U. But $U : [0, T] \times X \to K(V)$, V is a separable Banach space.*

  [2013 Liu ZH, Journal of Differential Equations, Accepted.]

Theorem 5.1. *Assume that the hypotheses $H(1) - H(7), H(L), H(\varepsilon)$ hold. Then Lagrange problem (P) admits at least one optimal control pair.*

6 Open Problems

- *Extension of the results to fractional evolutionary variational inequalities;*
- *Extension of the results to various classes of fractional hemivariational and variational-hemivariational inequalities;*
- *Optimal control problems, optimal shape design problems, controllability in contact mechanics.*

Thank you！

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