Numerical-analytic technique and parametrization for some nonlinear boundary value problems

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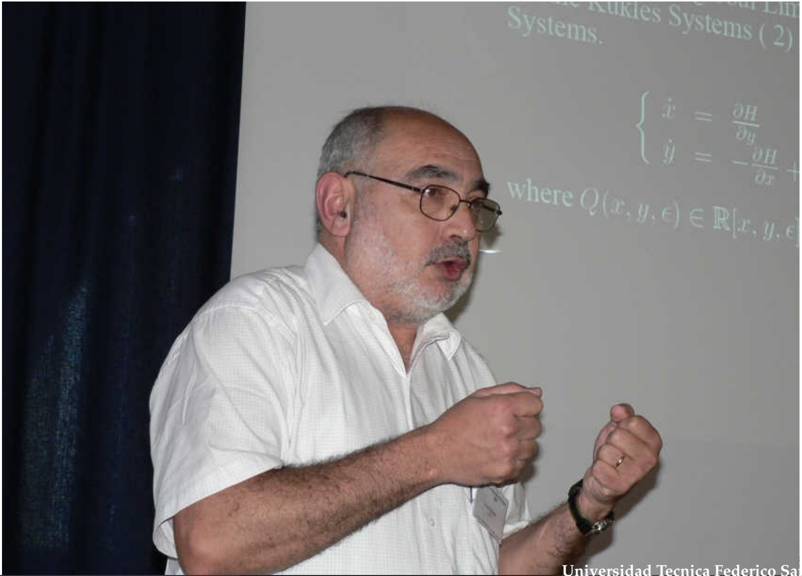
Dedicated to Professor Iván Szantó on the ocassion of his 60th birthday

HAPPY BIRTHDAY!!!



Abstract.

We show how a suitable parametrization technique can help in the numerical -analytic method based upon successive approximations for the investigation of solutions of non-linear boundary value problems with "inconvenient " boundary conditions defined by singular matrices, by integral terms or by nonlinear two-point restrictions.

Introduction.

Motivation of the interest to the theory of boundary value problems (BVPs) :

-wide application

-many unsolved (or partially solved) problems.

In this lecture I am going to speak about the BVPs only in the case of ODEs.

Various classes of BVPs are defined by :

-the type and the form of the given DE and

-the type of the boundary conditions.

In the theory of BVPs one distinguish :

-Linear BVPs

-Nonlinear BVPs.

General form of linear BVP:

$$\frac{dx}{dt} = A(t)x + f(t), \quad t \in [0, T],$$
$$l(x) = 0,$$

where

$$A \in C([0,T] \to \mathbb{R}^{n\times n}), \quad f \in C([0,T] \to \mathbb{R}^n),$$

Linear functional $\quad l : C^1(0,T) \to \mathbb{R}^n.$

Special cases.

Two-point linear BVP:

$$\frac{dx}{dt} = A(t)x + f(t), \ \ t \in [0, T],$$

$$B_0 x(0) + B_1 x(T) = d, \quad B_0, B_1 \in \mathbb{R}^{n\times n}, \ \ d \in \mathbb{R}^n.$$

Multipoint linear BVP :

$$\frac{dx}{dt} = A(t)x + f(t), \ \ t \in [0, T],$$

$$\sum_{j=0}^{m} B_j x(\tau_j) = d, \quad B_j \in \mathbb{R}^{n\times n}, \ \ d \in \mathbb{R}^n, \ 0 = \tau_0 < \tau_1 < \ldots < \tau_m = T$$

Linear BVPs theoretically can be reduced to the equivalent initial value problem:

$$\frac{dx}{dt} = A(t)x + f(t), \quad t \in [0, T],$$

$$x(0) = x_0, \quad x_0 = ?$$

By using

$$x(t, 0, x_0) = X(t)X^{-1}(0)x_0 + X(t)\int_0^t X^{-1}(s)f(s)ds,$$

In the case of multipoint linear BVP we obtain the linear algebraic system :

$$\left[B_0 + \sum_{j=1}^{m} B_j X\left(\tau_j\right)\right] x_0 = d - \sum_{j=1}^{m} B_j X\left(\tau_j\right) \int\limits_0^{\tau_j} X^{-1}(s) f(s) ds,$$

If

$$\det\left[B_0 + \sum_{j=1}^{m} B_j X\left(\tau_j\right)\right] \neq 0, \text{ we obtain } x_0.$$

Generalization for non-linear Cauchy problem.

Using the "shift" or "translation operator"
M.A.Krasnosel'skij , M., (1966,333p.) "Translation operators...."

$$\frac{dx}{dt} = f(t,x), \quad x(0) = x_0, \ (t,x) \in (-\infty,\infty) \times D, \quad D \subset \mathbb{R}^n$$

If $\exists!$ Solution with $x_0 \in D_H \subset D$:

$$x(t) = x(t, 0, x_0),$$

"Using the "shift" operator $U(\tau, 0, x_0)$ we can write :

$$x(\tau) = U(\tau, 0, x_0)x_0.$$

For example, in the case of $T-$ periodic solutions and $T-$ periodic boundary conditions:

$$x(0) = x(T),$$

for finding the initial value x_0 corresponding to the $T-$ periodic solution we should solve the following operator equation:

$$x_0 = U(T, 0, x_0)x_0.$$

By Newton method:

$$\left[I - U'(T, 0, x_0^{(k)})\right] \Delta x_0^{(k)} = x_0^{(k)} - x^{(k)}(T),$$

$$x_0^{(k+1)} = x_0^{(k)} + \Delta x_0^{(k)}, k = 0, 1, ...$$

where $U'(T, 0, x_0^{(k)})$ is a Frechet derivative of the "shift" operator $U(T, 0, x_0^{(k)})$ at the point $x_0^{(k)}$ and $x^{(k)}(t)$ is a solution of the following initial value problem

$$\frac{dx}{dt} = f(t, x), \quad x(0) = x_0^{(k)},$$

i.e.

$$x^{(k)}(t) = U(t, 0, x_0^{(k)}) x_0^{(k)}.$$

It is known, that

$$U'(T, 0, x_0^{(k)}) = X(T, x_0^{(k)}),$$

where

$$X(T, x_0^{(k)})$$

is a normal fundamental matrix at the point $t = T$ of the linear system of DEs:

$$\frac{dx}{dt} = A^{(k)}(t)x,$$

where

$$A^{(k)}(t) = \left(\frac{\partial f(t, x^{(k)}(t))}{\partial x} \right)$$

is a Jacobi matrix for $x = x^{(k)}(t)$.

Nonlinear BVPs.

We will study now various type BVPs for the nonlinear system of differential equations using some so called numerical-analytic techniques based on successive approximations :

$$\frac{dx}{dt} = f(t, x), \qquad (1)$$

where

$$f : [0, T] \times D \to \mathbb{R}^n,$$

$D \subset \mathbb{R}^n$ is a closed bounded domain.

Various types of boundary conditions :

Linear not separated :

$$Ax(0) + Cx(T) = d, \ \ A, C \in \mathbb{R}^{n \times n}, d \in \mathbb{R}^n,$$

$$\det C \neq 0 \ \ or \ \ \det C = 0,$$

$$Ax(0) + A_1 x(t_1) + A_2(t) x(t_2) + ... A_p x(t_p) + Cx(T) = d,$$

$$0 < t_1 < t_2 < ... < t_p < T.$$

Non-linear conditions:

$$g(x(0), x(T)) = 0, \ \ g : D \times D \to \mathbb{R}^n,$$

$$g\left(x(0), x(t_1), x(t_1), ..., x(t_p), x(T)\right) = 0.$$

Cauchy-Nicoletti conditions:

$$x_i(t_i) = d_i, i =, 2, ..., n,$$

$$0 = t_1 \leq t_2 \leq ... \leq t_n = T.$$

Example.

$n = 3$

$$x_1(t_1) = h_1, x_2(t_2) = h_2,\ x_3(t_3) = h_3,$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1(t_2) \\ x_2(t_2) \\ x_3(t_2) \end{bmatrix} +$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(T) \\ x_2(T) \\ x_3(T) \end{bmatrix} = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \end{bmatrix}.$$

Interpolation type conditions (Valle- Poisson condition):
$x_j(t_i) = d_i,\ i = 1, 2, ..., n,\ \ j \in \{1, 2, ..., n\}, 0 = t_1 < t_2 < ...t_n = T.$

Example $n = 5$,

$$x_1(t_1) = d_1, x_1(t_2) = d_2, x_2(t_3) = d_3, x_3(t_4) = d_4, x_3(t_5) = d_5.$$

Integral boundary conditions :

$$Ax(0) + \int_0^T B(s)x(s)ds + Cx(T) = d,$$

$$g\left(x(0), \int_0^T B(s)x(s)ds, x(T)\right) = d.$$

Problem setting:

The problem is to find the solution of the system of DEs (1) satisfying some one of above mentioned boundary conditions in a class of continuously differentiable functions :

$$x : [0, T] \to D, \quad x \in C^1([0, T], \mathbb{R}^n).$$

The analysis of the literature, devoted to the theory of non-linear boundary values problems, shows that various, so-called:

-analytic

- functional-analytic, (according to M.Farkas : perturbations methods)

- numerical and

- numerical-analytic methods based upon successive approximations

are now extensively studied.

Naturally, each group of methods has certain advantages and disadvantages.

The *analytic* methods in the theory of boundary value problems are generally used for qualitative investigation (existence, uniqueness, stability, dichotomy, reducibility, branching). See, e.g., [1, 2, 4, 5, 6, 7] and also the references in [8].

The group of *functional-analytic* methods for obtaining existence results widely uses the techniques of functional analysis, topological degree and the theory of approximate methods for solving operator equations [9, 10, 11, 12, 13, 14, 15, 16, 17, 18],[19], [20].

The group of *numerical* methods under the assumption of the existence of solutions gives practical numerical algorithms for approximate construction of solutions of given boundary-value problems [21, 22].

Note that numerical construction of approximate solutions is usually based on the idea of shooting method and may face certain difficulties because the regularity conditions for the right- hand side function , e.g. the Lipschitz condition, as a rule should be assumed globally, i.e. fulfilled for all the values of space variables in $\mathbb{R}^n$, which is quite often not the case.

Therefore, using the numerical methods a rigorous investigation of the solutions of boundary value problems tends to avoid the case when the range of the solution is restricted to a certain given bounded closed domain $D \subset \mathbb{R}^n$.

Example.

Let us consider the Cauchy problem

$$\frac{dx}{dt} = x^2 = f(t,x),$$
$$x(t_0) = x_0 \neq 0.$$

Obviously, the right hand side function satisfy the Lipschitz condition only locally.
The solution of the initial value problem

$$x(t) = -\frac{1}{t - (\frac{1}{x_0} + t_0)}$$

exists only on the interval $\left[t_0, t_0 + \frac{1}{x_0}\right)$.
Therefore, even the boundary value problem setting on the interval $[t_{0,}T]$ is impossible, when

$$T > t_0 + \frac{1}{x_0}.$$

We understand under the *numerical-analytic methods* the methods which enable one to represent the required solution in an analytic form, although some of its parameters or coefficients should be determined numerically by solving the system of algebraic or transcendental equation.

Among the *numerical-analytic methods* we share out those which are based upon successive approximations and for which the dimension of the appearing system of algebraic or transcendental equations coincides with the dimension of the given system of differential equations.

In the theory of nonlinear oscillations such types of numerical-analytic methods based upon successive approximations were apparently first developed by L. Cesari [23], J. Hale [12], and A. M. Samoilenko [24, 25].

The last approach was developed successfully later for general types of boundary-value problems in the books [26, 27, 28] and, e.g., the papers [29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 51, 52, 53, 54], [45], [46], [47], [50],[48],[49],[63],[67],[62],[65],[66],[64],[68].

Methods of the *numerical-analytic type*, in s sence, combine, advantages of the mentioned above approaches and are usually based upon certain iteration processes constructed explicity in analytic form.

Such an approach belongs to the few of them that offer constructive possibilities both for the investigation of the existence of solution and it approximate construction.

For a boundary value problem, the numerical-analytic approach usually replaces the problem by the Cauchy problem for a suitably perturbed system containing some artificially introduced vector parameter z, which most often has the meaning of the initial value of the solution and the numerical value of which is to be determined later.

The solution of the Cauchy problem for the perturbed system is sought for in analytic form by successive approximations.

The functional ”perturbation term”, by which the modified equation differs from the original one, depends explicitly on the parameter z and generates a system of algebraic or transcendental ”determining equations” from which the numerical values of z should be found.

The solvability of the determining system, in turn, may be checked by studying some approximations that constructed explicitly.

It is clear that the complexity of the given equations and boundary conditions has an essential influence both on the possibility of an efficient construction of approximate solutions and the subsequent solvability analysis.

The aim of this lecture to show how a suitable parametrization technique can simplify the using of some numerical -analytic techniques when dealing with ”inconvenient” linear boundary conditions determined by singular matrices or when the given two-point restrictions are nonlinear.

Two-point non-linear BVP with linear boundary conditions.

Let us consider in more detail the following nonlinear BVP, when for the sake of simplicity the linear boundary conditions are given in the usual matrix-vector form

$$\frac{dx}{dt} = f(t,x), \; f : [0,T] \times D \to \mathbb{R}^n, \; D \subset \mathbb{R}^n \text{ is a closed bounded domain,} \tag{2}$$

$$Ax(0) + Cx(T) = d, \;\; A, C \in \mathbb{R}^{n \times n}, d \in \mathbb{R}^n, \det \mathbf{C} \neq \mathbf{0}, \tag{3}$$

where the function

$$f : [0,T] \times D \to \mathbb{R}^n$$

is continuous .

The problem is to find the solution of BVP (2), (3) in the class of continuously differentiable functions

$$x : [0,T] \to D, \quad x \in C^1([0,T], \mathbb{R}^n),$$

Remark 1.

We note that from (3) when $A = I$, $C = -I$ *and* $d = 0$, *we obtain the periodic boundary conditions*

$$x(0) = x(T).$$

For the given function f we define the vectors

$$\delta_D(f) := \frac{1}{2}\left[\max_{(t,x)\in[0,T]\times D} f(t,x) - \min_{(t,x)\in[0,T]\times D} f(t,x)\right], \quad \beta = \frac{T}{2}\delta_D(f) \qquad (4)$$

and for any non-negative vector $r \in \mathbb{R}^n$ we understand

$$B(u,r) := \{\xi \in \mathbb{R}^n : |\xi - u| \leq r\}$$

as the $r-$neighbourhood of the $u \in \mathbb{R}^n$.

In equality (4) and in similar relations presented in what follows, the operations max, min and the signs $|\cdot|, \geq, \leq +, -,$ col for the vectors are understood componentwise.

We note, that the vector $\delta_D(f)$ satisfy the inequality

$$\delta_D(f) \leq \max_{(t,x)\in[0,T]\times D} |f(t,x)|$$

and the equality

$$\frac{1}{2}\left[\max_{(t,x)\in[0,T]\times D} f(t,x) - \min_{(t,x)\in[0,T]\times D} f(t,x)\right] = \max_{(t,x)\in[0,T]\times D} |f(t,x)|$$

holds if and only if

$$\max_{(t,x)\in[0,T]\times D} f(t,x) = -\min_{(t,x)\in[0,T]\times D} f(t,x) = \max_{(t,x)\in[0,T]\times D} |f(t,x)|.$$

Assume that the following conditions are satisfied for the BVP (2), (3):

(I) the function f is continuous in the domain $[0, T] \times D$ and satisfies the Lipschitz condition of the form

$$|f(t, u) - f(t, v)| \leq K |u - v| , \tag{5}$$

for all fixed $t \in [0, T], \{u, v\} \in D$, where K is a constant matrix with nonnegative components.

(II) there exist a nonempty set

$$\varnothing \neq D_\beta \subset D \tag{6}$$

such that

$$D_\beta := \left\{ z \in D : B\left(z + \frac{t}{T}\left[C^{-1}d - \left(C^{-1}A + I \right) z \right], \beta \right) \right\} \subset D \tag{7}$$

for all $t \in [0, T]$.

It means that the set D_β contains such points $z \in D$ for which the point

$$z + \frac{t}{T}\left[C^{-1}d - \left(C^{-1}A + I \right) z \right]$$

belongs to the domain D together with their β neighborhood.

(III) the spectral radius $r(K)$ of the matrix K satisfy the inequality

$$r(K) < \frac{10}{3T}, \tag{8}$$

which means that the greatest eigenvalue of the matrix

$$Q = \frac{3T}{10}K \tag{9}$$

is less then one.

To study the solution of the BVP (2), (3) let us introduce the parametrized sequence of vector- functions $\{x_m(t,z,\}_{m=0}^{\infty}$ depending on parameter $z \in D_\beta$:

$$x_m(t,z) := z + \int_0^t f(s, x_{m-1}(s,z))\, ds - \frac{t}{T}\int_0^T f(s, x_{m-1}(s,z))\, ds + \\ + \frac{t}{T}\left[C^{-1}d - (C^{-1}A + I)z\right], \quad m = 1, 2, ..., \qquad (10)$$

$$x_0(t,z) = z + \frac{t}{T}\left[C^{-1}d - (C^{-1}A + I)z\right]. \qquad (11)$$

It is easy to see that, for all $m \geq 0$ and $z \in D_\beta$,(moreover $z \in \mathbb{R}^n$) the functions x_m satisfy the linear two-point conditions (3)

$$Ax_m(0, z) + Cx_m(T, z) = d$$

and the initial condition

$$x_m(0, z) = z.$$

We now show that the sequence (10) is uniformly convergence with respect to $t \in [0, T]$ and establish the relationship between its limit function and the solution of the original nonlinear BVP (2), (3) .

Theorem 1 *(on the uniform convergence of the sequence).*

Let the function $f : [0,T] \times D \to \mathbb{R}^n$ on the right hand side of the system of differential equations (2) and the boundary conditions (3) satisfy conditions (I)-(III).

Then, for all fixed $z \in D_\beta$ the following assertions are true :

1. All functions of the sequence (10) are continuously differentiable functions satisfying the boundary conditions (3):

$$Ax_m(0,z) + Cx_m(T,z) = d, \quad m = 0,1,2,...$$

2. The sequence of functions (10) converges uniformly in $t \in [0, T]$ as $m \to \infty$ to the limit function

$$x^*(t, z) = \lim_{m\to\infty} x_m(t, z).$$

3. The limit function x^* satisfies the initial condition

$$x^*(0, z) = z$$

and the boundary condition (3).

4. For all $t \in [0, T]$ the limit function x^* is a unique continuously differentiable solution of the integral equation

$$x(t) = z + \int_0^t f(s, x(s))\, ds - \frac{t}{T}\int_0^T f(s, x(s))\, ds + \frac{t}{T}\left[C^{-1}d - (C^{-1}A + I)z\right]$$

or of the equivalent Cauchy problem for a modified system of differential equations

$$\frac{dx}{dt} = f(t, x) + \Delta(z),\ x(0) = z, \tag{12}$$

where $\Delta : D_\beta \to \mathbb{R}^n$ is a mapping given by formula

$$\Delta(z) = \frac{1}{T}\left[C^{-1}d - (C^{-1}A + I)z\right] - \frac{1}{T}\int_0^T f(s, x(s))\, ds. \tag{13}$$

5. For all $t \in (0, T)$, the deviation of the limit function x^* from its mth approximation satisfies the estimate.

$$|x^*(t,z) - x_m(t,z)| \leq \frac{10}{9}\alpha_1(t)Q^m(I-Q)^{-1}\delta_D(f), \tag{14}$$

where

$$\alpha_1(t) = 2t(1 - \frac{t}{T}) \leq \frac{T}{2},$$

the matrix Q is given by (9) and vector $\delta_D(f)$ is defined by (4).

Proof.

We prove that the sequence of functions (10) is a Cauchy sequence in the Banach space $C([0,T],\mathbb{R}^n)$.
First, we show that

$$x_m(t,z) \in D \text{ for all } (t,z) \in [0,T] \times D_\beta \text{ and } m \geq 0.$$

For this we need in the sequence of functions :

$$\alpha_{m+1}(t) = \left(1 - \frac{t}{T}\right) \int_0^t \alpha_m(s)ds + \frac{t}{T} \int_t^T \alpha_m(s)ds, m = 0, 1, 2, ..., \tag{15}$$

$$\alpha_0(t) = 1.$$

In particular, we have

$$\alpha_1(t) = 2t(1 - \frac{t}{T}). \tag{16}$$

Lemma 1 *(on the property of the sequence).*

Let the sequence of functions $\{\alpha_m\}_{m=0}^{\infty} \subset C([0,T], \mathbb{R})$ *be given by formula (15).*

Then :

(1) The function α_m *is symmetric with respect to the point* $\frac{T}{2}$ *for all* $m \geq 0$*, i.e.,*

$$\alpha_m(t) = \alpha_m(T-t), \ t \in [0,T]$$

$$\alpha_m(\frac{T}{2}-t) = \alpha_m(\frac{T}{2}+t), t \in \left[0, \frac{T}{2}\right].$$

(2) Sequence (15) can be represented alternatively as

$$\alpha_{m+1}(t) = \int_0^t \alpha_m(s)ds + \frac{t}{T}\int_t^{T-t} \alpha_m(s)ds =$$

$$= \frac{t}{T}\int_t^{T-t} \alpha_m(s)ds + \left(1 - \frac{t}{T}\right)\int_{T-t}^{t} \alpha_m(s)ds =$$

$$= \int_0^{T-t} \alpha_m(s)ds + \left(1 - \frac{t}{T}\right)\int_{T-t}^{t} \alpha_m(s)ds,\ t \in [0, T].$$

(3)For any $m \geq 1, \alpha_m(0) = \alpha_m(T)$ and $\alpha_m(t) > 0$ for all $t \in (0, T)$.

(4) The maximal value of every $\alpha_m(t), m \geq 0$, is achived at the point $\frac{T}{2}$, namely

$$\max_{t\in[0,T]} \alpha_m(t) = \alpha_m(\frac{T}{2}) = T^m \cdot \alpha_m^{(T=1)}\left(\frac{1}{2}\right),$$

where $\alpha_m^{(T=1)}(t)$ stands for $\alpha_m(t)$ corresponding to $T = 1$.

$$\alpha_m^{(T=1)}(t) = (1-t)\int_0^t \alpha_m(s)ds + t\int_t^1 \alpha_m(s)ds, m = 0, 1, 2, ...,$$

(5) For every $m \geq 1$

$$\frac{d\alpha_m(t)}{dt}.sign(t - \frac{T}{2}) \leq 0, \quad t \in [0, T],$$

that is, the function α_m is increasing on $\left(0, \frac{T}{2}\right)$ and decreasing on $\left(\frac{T}{2}, T\right)$.

Lemma 2 *(estimate with integral average).*

For an arbitrary continuous function $x : [0, T] \to \mathbb{R}$*, the estimate*

$$\left| \int_0^t \left[x(\tau) - \frac{1}{T} \int_0^T x(s) ds \right] d\tau \right| \leq \frac{1}{2} \alpha_1(t) \left[\max_{s \in [0,T]} x(s) - \min_{s \in [0,T]} x(s) \right]$$

is true for all $t \in [0, T]$ *, where* α_1 *is the function defined by equality (16).*

Lemma 3 *(estimate for the functions of the sequence).*

The sequence of functions $\alpha_m, m \geq 1$*,given by relation (15) satisfies the inequalities*

$$\alpha_{m+1}(t) \leq \frac{3T}{10}\alpha_m(t),\ m \geq 2;$$

$$\alpha_{m+1}(t) \leq \frac{10}{9}\left(\frac{3T}{10}\right)^m \alpha_1(t), m \geq 0.$$

Using these Lemmas from (10) for $m = 0$ we get the following relation

$$|x_1(t,z) - x_0(t,z)| = \left| \int_0^t \left[f(\tau, x_0(\tau, z)) - \frac{1}{T} \int_0^T f(s, x_0(s,z))ds \right] d\tau \right| \leq$$

$$\leq \frac{1}{2}\alpha_1(t) \left[\max_{(t,x)\in[0,T]\times D} f(t,x) - \min_{(t,x)\in[0,T]\times D} f(t,x) \right] \leq$$

$$\leq \alpha_1(t)\delta_D(f) \leq \frac{T}{2}\delta_D(f) = \beta. \quad (17)$$

So we conclude that $x_1(t,z) \in D$ for $t \in [0,T]$, $z \in D_\beta$.

By induction, one can show that all functions , $x_m(t,z)$, $m = 1, 2, ...$ also belong to the set D for $(t,z) \in [0,T] \times D_\beta$.

Consider the difference

$$x_{m+1}(t,z) - x_m(t,z) = \int_0^t \left[f\left(s, x_m\left(s,z\right)\right) - f\left(s, x_{m-1}\left(s,z\right)\right)\right] ds -$$

$$-\frac{t}{T}\int_0^T \left[f\left(s, x_m\left(s,z\right)\right) - f\left(s, x_{m-1}\left(s,z\right)\right)\right] ds, \ m = 1, 2, ...$$

We denote

$$r_m\left(t,z\right) = \left| x_m(t,z) - x_{m-1}(t,z) \right|, m = 1, 2, ...$$

By using the estimates of Lemmas and taking into account the Lipschitz condition , we find

$$r_{m+1}(t,z) \leq K\left[\left(1-\frac{t}{T}\right)\int_0^t r_m\,(s,z)\,ds+\frac{t}{T}\int_0^T r_m\,(s,z)\,ds\right]. \tag{18}$$

Thus by using (17), we obtain

$$r_1\,(t,z) \le \alpha_1(t)\delta_D(f). \tag{19}$$

On the base of (18) and (19) , we have

$$r_2\,(t,z) \le K\alpha_2(t)\delta_D(f).$$

By induction, it is possible to to show that

$$r_{m+1}(t,z) \le K^m\alpha_{m+1}(t)$$

and using the estimate of Lemma 3, we obtain

$$r_{m+1}(t,z) \le \frac{10}{9}\alpha_1(t)Q^m\delta_D(f), m = 1,2,... \tag{20}$$

By using inequality (20), we consider the difference and obtain the estimate

$$\left|x_{m+j}(t,z) - x_m(t,z)\right| \leq \sum_{i=1}^{j} r_{m+i}(t,z) \leq \frac{10}{9}\alpha_1(t)Q^m \sum_{i=0}^{j-1} Q^i \delta_D(f) \tag{21}$$

$$\leq \frac{10}{9}\alpha_1(t)Q^m (I-Q)^{-1}\delta_D(f).$$

Hence, inequality (21) enables us conclude that, according to the Cauchy criterion, the sequence $\{x_m\}$ uniformly converges on the set

$$[0, T] \times D_\beta$$

to a certain limit function x^*.

Since the functions x_m satisfy the boundary conditions (3) for arbitrary values of parameters, x^* also satisfies these conditions. Passing to the limit as $m \to \infty$ in relation (10), we conclude that the limit function satisfies the corresponding integral equation and, hence, is a solution of the Cauchy problem (12) , where Δ is the mapping defined by relation (13).Estimate (14) is a direct corollary of inequality (20).
Theorem is proved.

Relationship between the limit function and the solution of the BVP.

Parallel with the system (2), we consider the following equation with constant perturbation μ in the right-hand side :

$$\frac{dx}{dt} = f(t, x) + \mu \,, t \in [0T] \,,\ \mu \in \mathbb{R}^n, \quad \text{(control parameter)} \tag{22}$$

with the initial conditions

$$x(0) = z. \tag{23}$$

We show that, for all fixed $z \in D_\beta$ the parameter μ can be chosen so that the solution

$$x = x(t, z, \mu)$$

of the Cauchy problem (22), (23) will satisfy at the same time the linear boundary conditions (3).

Theorem 2 *(sufficient and necessary conditions for the existence of the unique control parameter).*

Assume the conditions of Theorem 1 are held. The solution $x = x(t, z, \mu)$ of the initial value problem will satisfy the boundary conditions

$$Ax(0, z, \mu) + Cx(T, z, \mu) = d \tag{24}$$

if and only if

$$\mu = \mu_z = \frac{1}{T}\left[C^{-1}d - (C^{-1}A + I)z\right] - \frac{1}{T}\int_0^T f\left(s, x^*\left(s, z\right)\right) ds. \tag{25}$$

Moreover,

$$x(t, z, \mu) = x^*\left(t, z\right), \tag{26}$$

$$x^*\left(t, z\right) = \lim_{m\to\infty} x_m(t, z).$$

Proof.

Sufficiency follows from Theorem 1. For necessity - indirect method.

Theorem 3 *(connection of the parametrized limit function $x^*(t,z)$ to the solution of original two-point BVP).*

Assume the conditions of Theorem 1.
Then the limit function

$$x^*(t,z) = \lim_{m\to\infty} x_m(t,z)$$

will be the solution of the original BVP (2) , (3) if and only if when the parameter

$$z = z^*,$$

where z^* is a solution of the system of n algebraic equations (determining system)

$$\frac{1}{T}\left[C^{-1}d - (C^{-1}A + I)z\right] - \frac{1}{T}\int_0^T f(s, x^*(s,z))\,ds = 0. \tag{27}$$

Proof.

It is sufficient to apply Theorem 2. We note that z^*, at the same time, is an initial value of the solution of the BVP at the point $t = 0$.

The following assertion shows that the dertermining system "catch" all the initial values of the possible solutions.

Lemma 4 *(on the zero points of the determining function).*

Assume that the conditions of Theorem 1 are satisfied and there exist some vector $z = z^*$, satisfying the system of determining equations (27). Then the non-linear BVP (2),(3) possesses a solution $x(\cdot)$ such that

$$x(0) = z^*.$$

Moreover, this solution has the form

$$x(t) = x^*\left(t, z^*\right) = \lim_{m\to\infty} x_m(t, z^*),\ t \in [0, T]\,. \tag{28}$$

Conversely, if the BVP (2),(3) has a solution $x(\cdot)$,then this solution necessarily has the form (28) with $z^* = x(0)$ and the system of determining equations is satisfied for

$$z = z^* = x\,(0)\,. \tag{29}$$

Proof.

On the base of Theorems above.

Sufficient conditions for the existence of solutions.

To investigate the solvability of the BVP (2), (3) side by side with the exact determining system

$$\Delta(z) = \frac{1}{T}\left[C^{-1}d - (C^{-1}A + I)z\right] - \frac{1}{T}\int_0^T f\left(s, x^*\left(s, z\right)\right) ds = 0, \tag{30}$$

we introduce an approximate determining system

$$\Delta_m(z) = \frac{1}{T}\left[C^{-1}d - (C^{-1}A + I)z\right] - \frac{1}{T}\int_0^T f\left(s, x_m\left(s, z\right)\right) ds = 0, \tag{31}$$

where $x_m\left(\cdot, z\right)$ is the vector -function defined by sequence (10) in explicit form.

Lemma 5 *(on the continuity of the exact limit function with respect to parameter).*

We need in the following Lemma.

Assume that the conditions of Theorem 1 are satisfied. Then the limit function $x^*(\cdot, z)$ satisfies with respect to second variable the Lipschitz condition of the form

$$\left| x^*(\cdot, z_1) - x_m(\cdot, z_2) \right| \leq \left[I + \frac{10}{9} K (I - Q)^{-1} \alpha_1(t) \right] R \left| z_1 - z_2 \right|, \qquad (32)$$

where

$$R := \max_{t \in [0,T]} \left| I - \frac{t}{T} \left(C^{-1} A + I \right) \right|.$$

Proof.

From (10) for $m = 1$

$$x_1(t, z_1) - x_1(t, z_2) = (z_1 - z_2) + \left(1 - \frac{t}{T}\right) \int_0^t \left[f(s, x_0(s, z_1)) - f(s, x_0(s, z_2))\right] ds +$$

$$- \frac{t}{T} \int_t^T \left[f(s, x_0(s, z_1)) - f(s, x_0(s, z_2))\right] ds +$$

$$+ \frac{t}{T} \left[\left(C^{-1}A + I\right)(z_2 - z_1)\right], \tag{33}$$

where

$$x_0(t, z_i) = z_i + \frac{t}{T}\left[C^{-1}d - \left(C^{-1}A + I\right) z_i\right]. \tag{34}$$

Using Lipschitz condition and the reprezentations (34), (15)

$$|x_1(t,z_1) - x_1(t,z_2)| \leq$$

$$\leq R|z_1 - z_2| + K\left[(1-\frac{t}{T})\int_0^t |x_0(s,z_1) - x_0(s,z_2)|\,ds + \right.$$

$$\left. + \frac{t}{T}\int_t^T |x_0(s,z_1) - x_0(s,z_2)|\,ds\right] \leq$$

$$\leq R|z_1 - z_2| + KR\left[(1-\frac{t}{T})\int_0^t ds + \frac{t}{T}\int_t^T ds\right]|z_1 - z_2| \leq$$

$$\leq [R + KR\alpha_1(t)]\,|z_1 - z_2|\,.$$

Similarly we get

$$|x_2(t,z_1) - x_2(t,z_2)| \leq$$

$$\leq \left[R + KR\alpha_1(t) + K^2R\alpha_2(t)\right]|z_1 - z_2|\,.$$

By induction , we find

$$|x_m(t,z_1) - x_m(t,z_2)| \leq \big[R + KR\alpha_1(t) + K^2R\alpha_2(t) + K^3R\alpha_3(t) + \ldots + \\ + K^mR\alpha_m(t)\big]\,|z_1 - z_2|\,. \quad (35)$$

By using inequality of Lemma 3 and the property of the matrix

$$Q = \frac{3T}{10}K$$

from (35) we get

$$|x_m(t,z_1) - x_m(t,z_2)| \leq \left[I + \frac{10}{9}K\alpha_1(t)\sum_{i=0}^{m-1}Q_i\right]R\,|z_1 - z_2|\,. \quad (36)$$

Passing in inequality (36) to the limit as $m \to \infty$, we conclude that

$$|x^*(t, z_1) - x^*(t, z_2)| \leq \left[I + \frac{10}{9} K (I - Q)^{-1} \alpha_1(t) \right] R \, |z_1 - z_2| \, .$$

The Lemma is proved.

Lemma 6 (*on the continuity of the exact determining function with respect to parameter*).

Assume that the conditions of Theorem 1 are satisfied. Then the function $\Delta(z)$ of the form is defined, continuous in the domain D_β and for all $z_1, z_2 \in D_\beta$ the following estimate is true for the deviation of functions

$$\Delta(z_i) = \frac{1}{T}\left[C^{-1}d - (C^{-1}A + I)z_i\right] - \\ - \frac{1}{T}\int_0^T f\left(s, x^*\left(s, z_i\right)\right) ds, z_i \in D_\beta, \quad i = 1, 2, \qquad (37)$$

$$|\Delta(z_1 - \Delta(z_2)| \leq \frac{1}{T}\left|C^{-1}A + I\right| |z_1 - z_2| + \\ + K\left[I + \frac{10T}{27}K(I - Q)^{-1}\right] R\,|z_1 - z_2|. \qquad (38)$$

Proof.

For all $z \in D_\beta$, there exist the limit of the uniformly convergent sequence of functions $\{x_m(t,z)\}$, which is also continuous . Therefore, if z varies in the domain D_β , the function $\Delta(z)$ is also continuous and bounded

$$\Delta(z) \leq \left| \frac{1}{T} \left[C^{-1}d - (C^{-1}A + I)z \right] \right| + M,$$

where

$$|f(t,z)| \leq M, \ (t,z) \in [0,T] \times M.$$

By using (37), the Lipschitz condition , (38) and taking into account that

$$\int_0^T \alpha_1(t)dt = \frac{T^2}{3},$$

we have

$$|\Delta(z_1 - \Delta(z_2)| \leq \frac{1}{T}\left|C^{-1}A + I\right| |z_1 - z_2| +$$

$$+\frac{1}{T}K\int_0^T \left[\left[I + \frac{10}{9}K(I-Q)^{-1}\alpha_1(t)\right] R\,|z_1 - z_2|\right] dt \leq$$

$$\leq \frac{1}{T}\left|C^{-1}A + I\right| |z_1 - z_2| + K\left[I + \frac{10T}{27}K(I-Q)^{-1}\right] R\,|z_1 - z_2|\,.$$

Lemma 7 *(on the difference between the exact and approximate determining functions).*

Assume that the conditions of Theorem 1 are satisfied. Then, for any $m \geq 1$, $z \in D_\beta$ for the deviation of exact and approximate determining functions given by (30), (31) we have the estimate

$$|\Delta(z) - \Delta_m(z)| \leq \frac{10T}{27} K Q^m (I - Q)^{-1} \delta_D(f). \tag{39}$$

Proof.

By direct computations, we find

$$|\Delta(z) - \Delta_m(z)| \leq \left| \frac{1}{T} \int_0^T f\left(s, x^*\left(s,z\right)\right) ds - \frac{1}{T} \int_0^T f\left(s, x_m\left(s,z\right)\right) ds \right| \leq$$

$$\leq \frac{1}{T} K \int_0^T \left| x^*\left(s,z\right) - x_m\left(s,z\right) \right| ds \leq \frac{1}{T} K \int_0^T \frac{10}{9} \alpha_1(s) Q^m (I-Q)^{-1} \delta_D(f) ds \leq$$

$$\leq \frac{10T}{27} K Q^m (I-Q)^{-1} \delta_D(f).$$

On the base of equations (30), (31) introduce the mappings

$$\Delta : \mathbb{R}^n \to \mathbb{R}^n, \tag{40}$$

$$\Delta_m : \mathbb{R}^n \to \mathbb{R}^n, \tag{41}$$

Definition 1 (*special ordering*).

Let $H \subset \mathbb{R}^n$ *be an arbitrary non-empty set.*
For any pair of functions

$$f_j = \operatorname{col}\left(f_{j1}(x), \ldots, f_{j,3n-q}(x)\right) : H \to \mathbb{R}^n,\ j = 1, 2$$

we write

$$f_1 \rhd_H f_2 \tag{42}$$

if and only if there exist a function $k : H \to \{1, 2, \ldots, n\}$ *such that*

$$f_{1,k(x)} > f_{2,k(x)},$$

for all $x \in H$*, which means that at least one of the components of the vector* f_1 *is greater then the corresponding component of the vector* f_2*.*

Let us introduce the certain bounded open set

$$\Omega \subset D_{\beta.} \tag{43}$$

Theorem 4 *(sufficient existence conditions).*

Assume that conditions of Theorem 1 hold and, moreover, one can specify an $m \geq 1$ *and a set* Ω *of form (43) such that on the boundary* $\partial\Omega$ *of domain* Ω

$$|\Delta_m| = \left| \frac{1}{T}\left[C^{-1}d - (C^{-1}A + I)z\right] - \frac{1}{T}\int_0^T f\left(s, x_m\left(s, z\right)\right) ds \right| \rhd_{\partial\Omega}$$

$$\rhd_{\partial\Omega} \frac{10T}{27} K Q^m (I - Q)^{-1} \delta_D(f) \qquad (44)$$

holds, where

$$\delta_D(f) := \frac{1}{2}\left[\max_{(t,x)\in[0,T]\times D} f\left(t, x\right) - \min_{(t,x)\in[0,T]\times D} f\left(t, x\right)\right].$$

If, in addition, the Brouwer degree of Δ_m over Ω with respect to zero satisfies the inequality

$$\deg(\Delta_m, \Omega, 0) \neq 0, \tag{45}$$

then there exist a value $z = z^* \in \Omega$ *such that the function*

$$x^*(t) = x^*(t, z^*) = \lim_{m \to \infty} x_m(t, z^*)$$

is a solution of the nonlinear boundary value problem (2), (3) with the initial value

$$x^*(0) = z^* \in \Omega \subset D_\beta. \tag{46}$$

Proof.

We show that the vector fields Δ and Δ_m are homotopic. For this introduce a family of vector mappings

$$P(\theta,z)=\Delta_m(z)+\theta\left[\Delta(z)-\Delta_m(z)\right],\; z\in\partial\Omega,\;\theta\in[0,1]\,. \tag{47}$$

It is clear that $P(\theta,z)$ is continuous on $\partial\Omega$ for any $\theta\in[0,1]$.
In addition

$$P(0,z)=\Delta_m(z),\;P(1,z)=\Delta(z),\text{ for all } z\in\partial\Omega.$$

For any $z \in \partial\Omega$ in view of of (47), we find

$$|P(\theta, z)| = |\Delta_m(z) + \theta [\Delta(z) - \Delta_m(z)]| \geq |\Delta_m(z)| - |\Delta(z) - \Delta_m(z)| \,. \quad (48)$$

On the other hand, by using the estimate (39) , we arrive at componentwise inequalities

$$|\Delta(z) - \Delta_m(z)| \leq \frac{10T}{27} K Q^m (I - Q)^{-1} \delta_D(f), \quad (49)$$

whence , in view of relations (44), (48),(49) , we conclude that

$$|P(\theta, z)| \rhd_{\partial\Omega} 0, \quad \theta \in [0, 1] \,.$$

It means , that transformation (47) is not degenerate and, hence, the vector fields Δ_m and Δ are homotopic.
By using relation (45) and the property of invariance of the Brouwer degree under homotopices, we conclude that

$$\deg(\Delta, \Omega, 0) = \deg(\Delta_m, \Omega, 0) \neq 0.$$

The classic topological result (e.g. see Theorem A.2.4 in [13]) guarantees the existence of a value

$$z = z^* \in \Omega$$

such that

$$\Delta(z^*) = 0,$$

i.e. z^* satisfies the system of determining equations.(27) and according to Theorem 3, we conclude that the function

$$x^*(t) = x^*\left(t, z^*\right) = \lim_{m\to\infty} x_m(t, z^*)$$

is a solution of the BVP (2), (3) with the initial condition

$$x^*(0) = z^* \in \Omega \subset D_\beta.$$

The Theorem is proved.

Necessary conditions for the solvability of the BVP.

Let us find necessary conditions for the solvability of the BVP (2), (3) , i.e. conditions necessary for a certain subdomain of the domain D_β

$$D_0 \subset D_\beta$$

to contain a point $z = z^*$ that determines the initial vale value $x^*(0)$ of the solution $x^*(t) = x^*(t, z^*) = \lim_{m \to \infty} x_m(t, z^*)$ of the BVP under conside-ration.

Theorem 5 *(necessary conditions for the existence).*

Suppose that the BVP (2), (3) satisfies all conditions of Theorem 1.
Then in order the subdomain $D_0 \subset D_\beta$ contain the point $z = z^*$ that determines the initial value $x^*(0)$ of the solution $x^*(t) = x^*(t, z^*) = \lim_{m\to\infty} x_m(t, z^*)$ of the BVP (2), (3) at $t = 0$, it is necessary that for all m and arbitrary $\bar{z} \in D_0$, the following inequality be satisfied:

$$|\Delta_m(\bar{z})| \leq \sup_{z\in D_0}\left\{\frac{1}{T}\left|C^{-1}A + I\right||z-\bar{z}| + \right.$$
$$\left. + K\left[I + \frac{10T}{27}K(I-Q)^{-1}\right]R\,|z-\bar{z}|\right\} +$$
$$+ \frac{10T}{27}KQ^m(I-Q)^{-1}\delta_D(f), \qquad (50)$$

where R is defined by (32).

Proof.

Let the integer m and $\bar{z} \in D_0$ be arbitrary.
Assume that the determining function $\Delta(z)$ of the form (30) vanishies at the point $z = z^*$,i.e. $\Delta(z^*) = 0$.
Let us use estimation (38) of Lemma 6 in the case where

$$z_1 = \bar{z}, z_2 = z^*.$$

Then it follows from (38) that

$$|\Delta(\bar{z})| \leq \frac{1}{T}\left|C^{-1}A + I\right| |\bar{z} - z^*| + K\left[I + \frac{10T}{27}K(I-Q)^{-1}\right] R\,|\bar{z} - z^*|. \quad (51)$$

In view of inequality (49), we get

$$|\Delta_m(\bar{z})| \leq |\Delta(\bar{z})| + \frac{10}{9}\alpha_1(t) Q^m (I-Q)^{-1}\delta_D(f). \quad (52)$$

The combination of (52) and (51) gives

$$|\Delta_m(\bar{z})| \leq \frac{10}{9}\alpha_1(t)Q^m(I-Q)^{-1}\delta_D(f) + \frac{1}{T}\left|C^{-1}A + I\right| |\bar{z} - z^*| + \\ + K\left[I + \frac{10T}{27}K(I-Q)^{-1}\right] R\,|\bar{z} - z^*| \quad (53)$$

which proves the validity of inequality (50).

Remark 2.

According to Theorem 5, we can indicate an algorithm for approximate determination of the initial values $x^*(0) = z^*$ of a solution of the given BVP.
For this purpose, we represent the set D_β as a union finitely many subsets D_i :

$$D_\beta = \cup_{i=1}^{N} D_i.$$

In every D, we choose an arbitrary point

$$z^i \in D_i$$

and for a certain number m, we calculate successive approximations $x_m\left(t, z^i\right)$ according to (10) and $\Delta_m(z^i)$ according to (31).

Then according to (50), we eliminate those subsets D_i for an arbitrary point of which the inverse inequality, i.e.

$$\left|\Delta_m(z^i)\right| > \sup_{z\in D_0}\left\{\frac{1}{T}\left|C^{-1}A+I\right|\left|z-z^i\right| + \right. \\ \left. + K\left[I+\frac{10T}{27}K(I-Q)^{-1}\right]R\left|z-z^i\right|\right\} + \\ + \frac{10T}{27}KQ^m(I-Q)^{-1}\delta_D(f) \quad (54)$$

is true, because by virtue of Theorem 5, such subsets cannot contain the point z^* that determines the initial value of a solution of the given BVP.

The other subsets D_i form a certain set

$$D_m^i,$$

which tends as $i, m \to \infty$ to the set $D(z^*)$ determining the initial values of a solution of the BVP (2), (3).
Any point $\widetilde{z} \in D_m^i$ can be regarded as an approximate value of the point z^*.

Parametrization for Reduction various "inconvenient" boundary conditions to the two-point linear restrictions.

1.**Two point boundary conditions with singular matrix :**

$$Ax(0) + Cx(T) = d, \quad A, C \in \mathbb{R}^{n\times n}, d \in \mathbb{R}^n, \tag{55}$$

$$\det C = 0, \tag{56}$$

$$C = \begin{bmatrix} C_{11} & C_{12} \\ 0_{21} & 0_{22} \end{bmatrix},$$

where $C_{11} : p \times p, \det C_{11} \neq 0;$ $C_{12} : p \times (n-p);$ $0_{21} : (n-p) \times p;$ $0_{22} : (n-p) \times (n-p);$

Parametrization :

$$x_{p+1}(T) = \lambda_{p+1}, x_{p+2}(T) = \lambda_{p+2}, ..., x_n(T) = \lambda_n. \tag{57}$$

New parametrized boundary conditions :

$$Ax(0) + C_1 x(T) = d + \lambda = d(\lambda), \tag{58}$$

where

$$C_1 = \begin{bmatrix} C_{11} & C_{12} \\ 0_{21} & I_{n-p} \end{bmatrix}, \ \det C_1 \neq 0.$$

$$\lambda = \mathrm{col}(0, 0, ..., 0, \lambda_{p+1}, \lambda_{p+2}, ..., \lambda_n). \tag{59}$$

2.**Two-point nonlinear boundary conditions:**

$$g\left(x\left(0\right), x\left(T\right)\right) = 0, \quad g : D \times D \to \mathbb{R}^n, \tag{60}$$

$$Ax(0) + Ix\left(T\right) + g\left(x\left(0\right), x\left(T\right)\right) = Ax(0) + Ix\left(T\right) \tag{61}$$

Parametrization **:**

$$z = x(0) = \mathrm{col}(x_1(0), ..., x_n(0)) = \mathrm{col}(z_1, z_2, ..., z_n) \tag{62}$$

$$\lambda = x(T) = \mathrm{col}(x_1(T), ..., x_n(T)) = \mathrm{col}(\lambda_1, \lambda_2, ..., \lambda_n) \tag{63}$$

$$Az + \lambda + g\left(z, \lambda\right) = d\left(z, \lambda\right) \tag{64}$$

Parametrized linear two-point boundary conditions :

$$Ax(0) + Ix\,(T) = d\,(z, \lambda) = Az + \lambda + g\,(z, \lambda) \tag{65}$$

Let us choose

$$A := 0$$

***Separated boundary conditions* :**

$$x(0) = z, \tag{66}$$

$$x(T) = d(z, \lambda) = \lambda + g\,(z, \lambda) \tag{67}$$

3. Three-point non-linear boundary conditions:

$$g\left(x\left(0\right), x(t_1,)x\left(T\right)\right) = 0, \;\; g : D \times D \times D \to \mathbb{R}^n, \; 0 < t_1 < T \tag{68}$$

Parametrization:

$$z = x(0) = \mathrm{col}(x_1(0), ..., x_n(0)) = \mathrm{col}(z_1, z_2, ..., z_n), \tag{69}$$

$$\eta = x(0) = \mathrm{col}(x_1(0), ..., x_n(0)) = \mathrm{col}(\eta_1, \eta_2, ..., \eta_n) \tag{70}$$

$$\lambda = x(T) = \mathrm{col}(x_1(T), ..., x_n(T)) = \mathrm{col}(\lambda_1, \lambda_2, ..., \lambda_n) \tag{71}$$

Parametrized linear two-point boundary conditions:

$$Ax(0) + x(T) = d(z, \eta, \lambda), \tag{72}$$

$$d(z, \eta, \lambda) = Az + \lambda + g(z, \eta, \lambda) \tag{73}$$

Let us choose

$$A := 0$$

Separated boundary conditions :

$$x(0) = z, \tag{74}$$

$$x(T) = d(z, \eta, \lambda) = \lambda + g(z, \eta, \lambda) \tag{75}$$

4. Integral boundary conditions of the form:

$$Ax(0) + Cx(T) + \int_0^T B(s)x(s)ds = d, \quad \det C = 0, \tag{76}$$

$$C = \begin{bmatrix} C_{11} & C_{12} \\ 0_{21} & 0_{22} \end{bmatrix},$$

where $C_{11} : p \times p, \det C_{11} \neq 0$; $C_{12} : p \times (n-p)$; $0_{21} : (n-p) \times p$; $0_{22} : (n-p) \times (n-p)$;

Parametrization:

$$z = x(0) = \mathrm{col}(x_1(0), ..., x_n(0)) = \mathrm{col}(z_1, z_2, ..., z_n) \tag{77}$$

$$\lambda = \int_0^T B(s)x(s)ds = \mathrm{col}\,(\lambda_1, ..., \lambda_n) \tag{78}$$

$$\eta = \mathrm{col}\left(0, 0, ..., 0, x_{p+1}(T), x_{p+2}(T), ..., x_n(T)\right) = \\ = \mathrm{col}(0, 0, ..., 0, \eta_{p+1}, \eta_{p+2}, \ldots, \eta_{n},) \tag{79}$$

Parametrized linear two-point boundary conditions:

$$Ax(0) + C_1 x(T) = d(\lambda, \eta) = d - \lambda + \eta \tag{80}$$

where

$$C_1 = \begin{bmatrix} C_{11} & C_{12} \\ 0_{21} & I_{n-p} \end{bmatrix}, \ \det C_1 \neq 0. \tag{81}$$

5. Integral boundary conditions of the form:

$$\int_0^T B(s)x(s)ds = d, \tag{82}$$

$$Ax(0) + x(T) + \int_0^T B(s)x(s)ds = d + Ax(0) + x(T)$$

Parametrization:

$$z = x(0) = \mathrm{col}(x_1(0), ..., x_n(0)) = \mathrm{col}(z_1, z_2, ..., z_n) \tag{83}$$

$$\lambda = \int_0^T B(s)x(s)ds = \mathrm{col}\,(\lambda_1, ..., \lambda_n) \tag{84}$$

$$\eta = \mathrm{col}(x_1(T), ..., x_n(T)) = \mathrm{col}(\eta_1, ..., \eta_n) \tag{85}$$

Parametrized linear two-point boundary conditions:

$$Ax(0) + x(T) = d + Az + \eta - \lambda = d(z, \lambda, \eta) \tag{86}$$

Separated boundary conditions if $A = 0$ **:**

$$x(0) = z \tag{87}$$

$$x(T) = d + \eta - \lambda = d(\lambda, \eta) \tag{88}$$

Two-point nonlinear boundary conditions.

Let us consider in more detail the boundary value problem of general form

$$\frac{dx}{dt} = f(t, x), \; t \in [0, T] \tag{89}$$

$$g(x(0), x(T)) = 0, \; g \in \mathbb{R}^n, \tag{90}$$

where

$$f : [0, T] \times D \to \mathbb{R}^n, \; g : D \times D \to \mathbb{R}^n$$

are continuous and $D \subset \mathbb{R}^n$ is the closure of a bounded domain.

Let us introduce the following parametrization :

$$z = x(0) = \mathrm{col}(z_1, ..., z_n),$$

$$\lambda = x(T) = \mathrm{col}(\lambda_1, ..., \lambda_n). \tag{91}$$

On the base of (91), instead of nonlinear boundary conditions (90) we can consider the linear separated two-point condition of form

$$x(0) = z, \; x(T) = d(z, \lambda), \tag{92}$$

where

$$d(z, \lambda) = \lambda + g\,(z, \lambda)\,.$$

We note that the boundary condition (92) is a special case of two-point linear non-separated parametrized boundary conditions of form

$$Ax(0) + Cx(T) = d(z, \lambda)$$

with zero matrix A and $C = I$.
Instead of boundary value problem (89), (90) we shall consider the problem (89), (92) with linear separated boundary conditions.

Remark 3.

It is easy to see that the solution of the original boundary value problem (89), (90) coincide with those solutions of the two-point boundary value problem (89), (92)) for which the additional conditions (91) are satisfied.

It is supposed that the following conditions hold:

(I) the function f is continuous in the domain $[0, T] \times D$ and satisfies the Lipschitz condition of the form

$$|f(t, u) - f(t, v)| \leq K |u - v|, \tag{93}$$

for all fixed $t \in [0, T], \{u, v\} \in D$, where K is a constant matrix with nonnegative components.

(II) there exist a nonempty set

$$\varnothing \neq D_\beta \subset D \tag{94}$$

such that

$$D_\beta := \left\{ z \in D : B\left(z + \frac{t}{T}(d\,(z,\lambda) - z)\,, \beta \right) \subset D,\ \forall\ \lambda \in D \text{ and } t \in [0,T] \right\} \tag{95}$$

for$\forall\ \lambda \in D$ and $t \in [0,T]$.
It means that the set D_β contains such points $z \in D$ for which the point

$$z + \frac{t}{T}\left[C^{-1} d(z,\lambda) - \left(C^{-1}A + I \right) z \right] \tag{96}$$

belongs to the domain D together with their β neighborhood.

(III) the spectral radius $r(K)$ of the matrix K satisfy the inequality

$$r(K) < \frac{10}{3T}, \tag{97}$$

which means that the greatest eigenvalue of the matrix

$$Q = \frac{3T}{10} K \tag{98}$$

is less then one.

To study the solution of the auxilary two-point boundary value problem (89), (92) let us introduce the parametrized sequence of functions $\{x_m(t,z,\lambda)\}_{m=0}^{\infty}$ depending on parameters

$$z = x(0) = \mathrm{col}(z_1, ..., z_n),$$

$$\lambda = x(T) = \mathrm{col}(\lambda_1, ..., \lambda_n).$$

$$x_m(t,z,\lambda) := z + \int_0^t f(s, x_{m-1}(s,z,\lambda))\,ds -$$

$$-\frac{t}{T}\int_0^T f(s, x_{m-1}(s,z,\lambda))\,ds + \frac{t}{T}\left[d(z,\lambda) - z\right], \; m = 1, 2, ..., \tag{99}$$

where

$$x_0(t,z,\lambda) = z + \frac{t}{T}\left[d(z,\lambda) - z\right], \;\; d(z,\lambda) = \lambda + g(z,\lambda). \tag{100}$$

Theorem 6 *(convergence of the sequence).*

The following statements are true.

Assume that the vector-function f satisfies conditions (I)-(III).
Then for all fixed $\lambda \in D$ and $z \in D_\beta$:
1.All the members of sequence (166) are continuously differentiable functions satisfying the parametrized boundary conditions (92) .
2. The sequence of functions (166) converges to a limit function $x^*(t, z, \lambda)$

$$x^*(t, z, \lambda) = \lim_{m \to \infty} x_m(t, z, \lambda) \tag{101}$$

uniformly in $t \in [0, T]$,
3.The limit function satisfies the parametrized two-point boundary conditions (92):

$$x^*(0, z, \lambda) = z, \; x^*(T, z, \lambda) = d(z, \lambda).$$

4.The limit function $x^*(t,z,\lambda)$ is a unique continuously differentiable solution of the integral equation

$$x(t) = z + \int_0^t f(s,x(s))ds - -\frac{t}{T}\int_0^T f(s,x(s))ds + \frac{t}{T}\left[d(z,\lambda) - z\right], \qquad (102)$$

or of the equivalent Cauchy problem for a modified system of differential equations

$$\frac{dx}{dt} = f(t,x) \ + \Delta(z,\lambda), \ x(0) = z \qquad (103)$$

where $\Delta(z,\lambda) : D_\beta \times D \to \mathbb{R}^n$ is a mapping given by formula

$$\Delta(z,\lambda) = \frac{1}{T}\left[d(z,\lambda) - z\right] - \frac{1}{T}\int_0^T f(s,x(s))ds. \qquad (104)$$

5.The following estimate holds.

$$|x^*(t,z,\lambda) - x_m(t,z,\lambda)| \leqslant$$

$$\leqslant \frac{10}{9}\alpha_1(t)Q^m(I-Q)^{-1}\delta_D(f), \quad t \in (0,T). \tag{105}$$

Theorem 7 *(connection of the parametrized limit function $x^*(t,z)$ to the solution of original two-point BVP).*

Assume the conditions of Theorem 6. Then the function

$$x^*(t, z^*, \lambda^*) = \lim_{m\to\infty} x_m(t, z^*, \lambda^*)$$

is a solution of the original boundary value problem (89),(90) with non-linear boundary conditions if and only if the pair (z^*, λ^*) satisfies the system of $n+n$ algebraic equations

$$\Delta(z,\lambda) = \frac{1}{T}\left[d(z,\lambda) - z\right] - \frac{1}{T}\int_0^T f(s, x^*(s,z,\lambda))ds = 0, \tag{106}$$

$$x^*(T, z, \lambda) = \lambda. \tag{107}$$

We note, that it is possible to prove the existence of a solution based on the properties of a certain approximation $x_m(t, z, \lambda)$ known in the analytic form by studying the "approximate determining system"

$$\Delta_m(z, \lambda) = \frac{1}{T}\left[d(z, \lambda) - z\right] - \frac{1}{T}\int_0^T f(s, x_m(s, z, \lambda))ds = 0, \tag{108}$$

$$x_m(T, z, \lambda) = \lambda. \tag{109}$$

Definition 2.

For any pair of indices i_1 *and* i_2 *between* 1 *and* n *,* $i_2 \geq i_1$*, define the* $(i_2 - i_1 + 1) \times n$ *dimension matrix*

$$J_{i_1,i_2} := (0_{i_2-i_1+1,i_1-1}, I_{i_2-i_1+1}, 0_{2-i_1+1,n-i_2}),$$

where $0_{i,j}$ *is the zero matrix of dimension* $i \times j$*,* I_k *is the unit matrix of dimension* k*,* $0_i = 0_{i,i}$*.*

It is easy to see that the left multiplication of a vector by a matrix J_{i_1,i_2} *, in fact, means the selection of its components with numbers from* i_1 *to* i_2*.*

On the base of equations (106), (107) and (108), (109) introduce the mappings

$$\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{2n},$$

$$\Phi_m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{2n},$$

by setting

$$\Phi(z,\lambda) = \left(\begin{array}{c} \Delta(z,\lambda) = \frac{1}{T}\left[d(z,\lambda) - z\right] - \frac{1}{T}\int\limits_0^T f(s, x^*(s,z,\lambda))ds \\ J_{1,n}\left(x^*(T,z,\lambda) - \lambda\right) \end{array} \right), \tag{110}$$

$$\Phi_m(z,\lambda) = \left(\begin{array}{c} \Delta_m(z,\lambda) = \frac{1}{T}\left[d(z,\lambda) - z\right] - \frac{1}{T}\int\limits_0^T f(s, x_m(s,z,\lambda))ds \\ J_{1,n}\ \left(x_m(T,z,\lambda) - \lambda\right) \end{array} \right). \tag{111}$$

for all $z \in D_\beta, \lambda \in D$.

Definition 3.

Let $H \subset \mathbb{R}^{2n}$ *be an arbitrary non-empty set. For any pair of functions*

$$f_j = \mathrm{col}\left(f_{j1}(x), ..., f_{j,3n-q}(x)\right) : H \to \mathbb{R}^{2n},\ j = 1, 2$$

we write

$$f_1 \rhd_H f_2 \tag{112}$$

if and only if there exist a function $k : H \to \{1, 2, ..., 2n\}$ *such that*

$$f_{1,k(x)} > f_{2,k(x)},$$

for all $x \in H$*, which means that at least one of the components of the vector* f_1 *is greater then the corresponding component of the vector* f_2.

Let us consider the domain

$$\Omega = D_1 \times \Lambda \subset \mathbb{R}^{2n} \tag{113}$$

where $D_1 \subset D_\beta$, $\Lambda \subset D$ are certain bounded open sets.

Theorem 8 *(sufficient existence conditions).*

Assume that conditions of Theorem 6 hold and, moreover, one can specify an $m \geq 1$ *and a set* Ω *of form (113) such that on the boundary* $\partial\Omega$ *of domain* Ω

$$|\Phi_m| \rhd_{\partial\Omega} \begin{pmatrix} \frac{10T}{27} K Q^m (I-Q)^{-1} \delta_D(f) \\ \frac{5T}{9} Q^m (I-Q)^{-1} \delta_D(f), \end{pmatrix} \tag{114}$$

holds, where

$$\delta_D(f) = \frac{1}{2} \left[\max_{(t,x)\in[0,T]\times D} f(t,x) - \min_{(t,x)\in[0,T]\times D} f(t,x) \right].$$

If, in addition, the Brouwer degree of Φ_m over Ω with respect to zero satisfies the inequality

$$\deg(\Phi_m, \Omega, 0) \neq 0, \tag{115}$$

then there exist a pair $(z^*, \lambda^*) \in \Omega$ such that the function

$$x^*(t) = x^*(t, z^*, \lambda^*) = \lim_{m \to \infty} x_m(t, z^*, \lambda^*), t \in [0, T] \tag{116}$$

is a solution of the nonlinear boundary value problem (89), (90) with the initial value

$$x^*(0) = z^* \in D_1 \subset D_\beta.$$

Integral boundary value problems.

We consider the BVP

$$\frac{dx}{dt} = f(t, x), \quad t \in [0, T], \tag{117}$$

$$Ax(0) + Cx(T) + \int_0^T B(s)x(s)ds = d, \quad \det C = 0, \tag{118}$$

where

$$f : [0, T] \times D \to \mathbb{R}^n,$$

$$D \subset \mathbb{R}^n \text{ is a closed bounded domain,}$$

$$C = \begin{bmatrix} C_{11} & C_{12} \\ 0_{21} & 0_{22} \end{bmatrix}, \tag{119}$$

$C_{11} : p \times p, \det C_{11} \neq 0$; $C_{12} : p \times (n-p)$; $0_{21} : (n-p) \times p$; $0_{22} : (n-p) \times (n-p)$.

The problem is to find the solution of the system of differential equations (117) satisfying the integral boundary restrictions (118) in a class of continuously differentiable vector functions $x. [0, T] \to D$.

Parametrization:

$$z = x(0) = \mathrm{col}(x_1(0), ..., x_n(0)) = \mathrm{col}(z_1, z_2, ..., z_n) \tag{120}$$

$$\lambda = \int_0^T B(s)x(s)ds = \mathrm{col}\,(\lambda_1, ..., \lambda_n) \tag{121}$$

$$\eta = \mathrm{col}\left(0, 0, ..., 0, x_{p+1}(T), x_{p+2}(T), ..., x_n(T)\right) = \\ \mathrm{col}(0, 0, ..., 0, \eta_{p+1,}\eta_{p+2,} \ldots, \eta_{n,}) \tag{122}$$

Parametrized linear two-point boundary conditions:

$$Ax(0) + C_1 x(T) = d(\lambda, \eta) = d - \lambda + \eta \tag{123}$$

where

$$C_1 = \begin{bmatrix} C_{11} & C_{12} \\ 0_{21} & I_{n-p} \end{bmatrix}, \ \det C_1 \neq 0. \tag{124}$$

So, instead of the integral BVP (117), (118) we study an equivalent parametrized one, containing already linear two-point boundary conditions with nonsingular matrixes:

$$\frac{dx}{dt} = f(t, x), \quad t \in [0, T], \tag{125}$$

$$Ax(0) + C_1 x(T) = d(\lambda, \eta) = d - \lambda + \eta \tag{126}$$

Remark 4.

The set of solutions of integral BVP (117), (118) coincides with the set of the solutions of the parametrized problem (125), (126), satisfying additional conditions (120), (121), (122).

Let us put :

$$D_0 := \left\{ \int_0^T B(s)x(s)ds : x \in C\left([0,T], D\right) \right\}. \tag{127}$$

It is supposed that the following conditions hold:

(I) the function f is continuous in the domain $[0, T] \times D$ and satisfies the Lipschitz condition of the form

$$|f(t,u) - f(t,v)| \leq K\,|u - v|\,, \tag{128}$$

for all fixed $t \in [0, T]$, $\{u, v\} \in D$, where K is a constant matrix with nonnegative components.

(II) there exist a nonempty set

$$\varnothing \neq D_\beta \subset D \tag{129}$$

such that

$$D_\beta := \left\{ z \in D : B\left(z + \frac{t}{T}\left[C_1^{-1} d\left(\lambda, \eta\right) - \left(C_1^{-1} A + I \right) z \right) \right], \beta \right) \subset D \right\}, \tag{130}$$

for$\forall$ $\lambda \in D_0, \eta \in D$ and $t \in [0, T]$,where

$$\beta = \frac{T}{2}\delta_D(f), \ \ \delta_D(f) := \frac{1}{2}\left[\max_{(t,x)\in[0,T]\times D} f(t,x) - \min_{(t,x)\in[0,T]\times D} f(t,x) \right], \tag{131}$$

It means that the set D_β contains such points $z \in D$ for which the point

$$z + \frac{t}{T}\left[C_1^{-1} d\left(\lambda, \eta\right) - \left(C_1^{-1} A + I \right) z \right) \right] \tag{132}$$

belongs to the domain D together with their β neighborhood.

(III) the spectral radius $r(K)$ of the matrix K satisfy the inequality

$$r(K) < \frac{10}{3T},$$

which means that the greatest eigenvalue of the matrix

$$Q = \frac{3T}{10}K \tag{133}$$

is less then one.

Let us connect with the parametrized BVP (125), (126) the sequence of functions

$$x_m(t,z,\lambda,\eta) := z + \int_0^t f(s, x_{m-1}(t,z,\lambda,\eta))\,ds - \frac{t}{T}\int_0^T f(s, x_{m-1}(t,z,\lambda,\eta))\,ds + \frac{t}{T}\left[C_1^{-1}d(\lambda,\eta) - (C_1^{-1}A + I)z\right], \quad m = 1,2,...., \tag{134}$$

where $m = 1, 2,...,$

$$x_0(t,z) = z + \frac{t}{T}\left[C_1^{-1}d(\lambda,\eta) - (C_1^{-1}A + I)z\right] \tag{135}$$

and z,λ, η are considered as parameters.

Theorem 9 (*on the uniform convergence of the sequence*).

Let the function $f : [0, T] \times D \to \mathbb{R}^n$ *on the right hand side of the system of differential equations (125) and the boundary conditions (126) satisfy conditions (I)-(III).*
Then, for all fixed $z \in D_\beta, \lambda \in D_0, \eta \in D$ *the following assertions are true :*
1. All functions of the sequence (134) are continuously differentiable functions satisfying the parametrized boundary conditions (126):

$$Ax_m(0, z, \lambda, \eta) + Cx_m(T, z, \lambda, \eta) = d(\lambda, \eta), \quad m = 0, 1, 2, \ldots$$

2.The sequence of functions (134) converges uniformly in $t \in [0, T]$ *as* $m \to \infty$ *to the limit function*

$$x^*(t, z, \lambda, \eta) = \lim_{m\to\infty} x_m(t, z, \lambda, \eta). \tag{136}$$

3.The limit function x^* *satisfies the initial condition* $x^*(0, z, \lambda, \eta) = z$ *and the*

4.For all $t \in [0, T]$ the limit function x^* is a unique continuously differentiable solution of the integral equation

$$x(t) = z + \int_0^t f(s, x(s))\, ds - \frac{t}{T} \int_0^T f(s, x(s))\, ds + \frac{t}{T} \left[C_1^{-1} d(\lambda, \eta) - (C_1^{-1} A + I) z \right] \tag{137}$$

or of the equivalent Cauchy problem for a modified system of differential equations

$$\frac{dx}{dt} = f(t, x) + \Delta(z, \lambda, \eta), \; x(0) = z, \tag{138}$$

where $\Delta : D_\beta \times D_0 \times D \to \mathbb{R}^n$ is a mapping given by formula

$$\Delta(z, \lambda, \eta) = \frac{1}{T} \left[C_1^{-1} d(\lambda, \eta) - (C_1^{-1} A + I) z \right] - \frac{1}{T} \int_0^T f(s, x(s))\, ds. \tag{139}$$

5.For all $t \in (0, T)$, the deviation of the limit function x^* from its mth approximation satisfies the estimate.

$$|x^*(t,z) - x_m(t,z)| \leq \frac{10}{9}\alpha_1(t)Q^m(I-Q)^{-1}\delta_D(f), \tag{140}$$

where

$$\alpha_1(t) = 2t(1-\frac{t}{T}) \leq \frac{T}{2}, \ \ Q = \frac{3T}{10}K,$$

$$\delta_D(f) := \frac{1}{2}\left[\max_{(t,x)\in[0,T]\times D} f(t,x) - \min_{(t,x)\in[0,T]\times D} f(t,x)\right].$$

Theorem 10 (*connection of the parametrized limit function $x^*(t, z, \lambda, \eta)$ to the solution of the integral BVP*).

Assume the conditions of Theorem 9.
Then the function

$$x^*(t, z^*, \lambda^*, \eta^*) = \lim_{m\to\infty} x_m(t, z^*, \lambda^*, \eta^*)$$

is a solution of the original integral boundary value problem (117),(118) if and only if the triplet

$$z^* = \mathrm{col}(z_1^*, ..., z_n^*),$$
$$\lambda^* = \mathrm{col}(\lambda_1^*, ..., \lambda_n^*), \tag{141}$$
$$\eta^* = \mathrm{col}(0, 0, ..., 0, \eta_{p+1}^*, \eta_{p+2}^*, ..., \eta_n^*))$$

satisfy the system of determining algebraic equations

$$\Delta(z,\lambda,\eta) := \frac{1}{T}\left[C_1^{-1}d\left(\lambda,\eta\right) - (C_1^{-1}A + I)z\right] - \\ - \frac{1}{T}\int\limits_0^T f\left(s, x^*\left(s,z,\lambda,\eta\right)\right)ds = 0, \quad (142)$$

$$V(z,\lambda,\eta) := \int\limits_0^T B(s)x^*\left(s,z,\lambda,\eta\right)ds - \lambda = 0 \quad (143)$$

$$J_{p+1,n}(x^*\left(T,z,\lambda,\eta\right) - \eta) = 0, \quad (144)$$

Althought Theorem 10 gives sufficient and necessary conditions for the solvability and construction of the solution of the given integral BVP, its application faces with difficulties due to the fact that the explicit form of the functions

$$\begin{aligned} \Delta &: D_\beta \times D_0 \times D \to \mathbb{R}^n, \\ V &: D_\beta \times D_0 \times D \to \mathbb{R}^n, \\ x^*(\cdot, z, \lambda, \eta) &= \lim_{m\to\infty} x_m(\cdot, z, \lambda, \eta) \end{aligned} \tag{145}$$

in (142), (143), (144) is usually unknown.

This complication can be overcome by using the properties of the function $x_m(\cdot, z, \lambda, \eta)$ of the form (134) for a fixed m , which will lead one instead of the exact determining system (142), (143), (144) to the mth approximate system of determining equations of the form :

$$\Delta_m(z,\lambda,\eta) := \frac{1}{T}\left[C_1^{-1}d\left(\lambda,\eta\right) - (C_1^{-1}A+I)z\right] - \\ - \frac{1}{T}\int_0^T f\left(s, x_m\left(s,z,\lambda,\eta\right)\right)ds = 0, \quad (146)$$

$$V(z,\lambda,\eta) := \int_0^T B(s)x_m\left(s,z,\lambda,\eta\right)ds - \lambda = 0 \quad (147)$$

$$J_{p+1,n}(x_m\left(T,z,\lambda,\eta\right) - \eta) = 0, \quad (148)$$

It is important to note that, unlike to system (142),(143),(144) the mth approximate determining system (146),(147), (148) contains only terms involving the function $x_m(\cdot, z, \lambda, \eta)$ and, thus known explicitly.

On the base of equations (142), (143), (144) and (146),(147), (148) let us introduce the mappings :

$$\Phi : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{3n},$$

$$\Phi_m : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{3n}$$

by setting for all z, λ, η of form (120), (121), (122) :

$$\Phi(z,\lambda,\eta) := \begin{pmatrix} \frac{1}{T}\left[C_1^{-1}d(\lambda,\eta) - (C_1^{-1}A + I)z\right] - \\ -\frac{1}{T}\int\limits_0^T f\left(s, x^*\left(s,z,\lambda,\eta\right)\right)ds \\ \int\limits_0^T B(s)x^*\left(s,z,\lambda,\eta\right)ds - \lambda \\ J_{p+1,n}(x^*\left(T,z,\lambda,\eta\right) - \eta) \end{pmatrix} \tag{149}$$

and

$$\Phi_m(z,\lambda,\eta) := \begin{pmatrix} \frac{1}{T}\left[C_1^{-1}d\left(\lambda,\eta\right) - (C_1^{-1}A + I)z\right] - \\ -\frac{1}{T}\int\limits_0^T f\left(s, x_m\left(s,z,\lambda,\eta\right)\right)ds \\ \int\limits_0^T B(s)x_m\left(s,z,\lambda,\eta\right)ds - \lambda \\ J_{p+1,n}(x_m\left(T,z,\lambda,\eta\right) - \eta) \end{pmatrix} \tag{150}$$

Let us consider the set

$$\Omega = D_1 \times \Lambda_1 \times D_2. \tag{151}$$

Theorem 11.

Assume the conditions of Theorem 9 and moreover, one can specify an $m \geq 1$ *a set* $\Omega \subset \mathbb{R}^{3n}$ *of the form (151) such that*

$$|\Phi_m| \triangleright_{\partial\Omega} \begin{pmatrix} \frac{10T}{27} K Q^m (I-Q)^{-1} \delta_D(f) \\ \frac{10}{9} \widetilde{B} Q^m (I-Q)^{-1} \delta_D(f) \\ \frac{5T}{9} Q^m (I-Q)^{-1} \delta_D(f) \end{pmatrix} \tag{152}$$

holds, where $\partial\Omega$ *is a bound of domain* Ω *and* $\widetilde{B} := \int_0^T |B(s)| \alpha_1(s) ds$.

If, in addition, the Brouwer degree of Φ_m over Ω with respect to zero satisfies the inequality

$$\deg(\Phi_m, \Omega, 0) \neq 0, \tag{153}$$

then there exist a triplet $(z^*, \lambda^*, \eta^*) \in \Omega$ such that the function

$$x^*(t) = x^*\left(t, z^*, \lambda^*, \eta^*\right) = \lim_{m \to \infty} x_m(t, z^*, \lambda^*, \eta^*) \tag{154}$$

is a solution of the given integral BVP with the initial value

$$x^*(0) = z^*.$$

Example of the nonlinear BVP with 3–point nonlinear boundary conditions.

Consider the system

$$\begin{cases} \frac{dx_1}{dt} = 0.05x_2 - 0.005t^2 + 0.1 = f_1(t, x_1, x_2), \\ \frac{dx_2}{dt} = -x_2^2 + 0.5x_1 + 0.01t^4 + 0.15t = f_2(t, x_1, x_2), \end{cases} \tag{155}$$

where $t \in \left[0, \frac{1}{2}\right]$,
with non–linear three–point boundary conditions

$$\begin{cases} g_1\left(x(0), x(\frac{1}{4}), x(1)\right) := x_1\left(\frac{1}{2}\right) + x_2^2(0) - x_1\left(\frac{1}{4}\right) - 0.025 = 0, \\ g_2(x(0), x(\frac{1}{4}), x(1)) := x_1(0) + x_2\left(\frac{1}{2}\right) - x_2(0) - 0.025 = 0. \end{cases} \tag{156}$$

It is easy to check that an exact solution of the problem (155), (156) are the functions

$$\begin{cases} x_1^* = 0.1t, \\ x_2^* = 0.1t^2. \end{cases} \tag{157}$$

Suppose that the boundary–value problem (155), (156) is considered in the domain

$$D = \{(x_1, x_2) : |x_1| \leq 0.42, |x_2| \leq 0.4\}. \tag{158}$$

Boundary conditions (156) can be rewritten in the form

$$Ax(0)+Cx\left(\frac{1}{2}\right)+g\left(x\left(0\right),x\left(\frac{1}{4}\right)x\left(\frac{1}{2}\right)\right)=Ax(0)+Cx\left(\frac{1}{2}\right), \quad (159)$$

where

$$A=\begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix},$$

$$C=I,$$

$$I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$g\left(x\left(0\right),x\left(\tfrac{1}{4}\right)x\left(\tfrac{1}{2}\right)\right)=col\left(g_1\left(x\left(0\right),x\left(\tfrac{1}{4}\right)x\left(\tfrac{1}{2}\right)\right),g_2\left(x\left(0\right),x\left(\tfrac{1}{4}\right)x\left(\tfrac{1}{2}\right)\right)\right).$$

Let us replace the values of the components of the solution of the boundary–value problem (155), (156) in the points $t = 0$, $t = \frac{1}{4}$ and $t = \frac{1}{2}$ by parameters z_1, z_2, η_1, η_2 and λ_1, λ_2 :

$$\begin{aligned} x(0) &= col\,(x_1(0), x_2(0)) = col\,(z_1, z_2), \\ x\left(\tfrac{1}{4}\right) &= col\left(x_1\left(\tfrac{1}{4}\right), x_2\left(\tfrac{1}{4}\right)\right) = col\,(\eta_1, \eta_2), \\ x\left(\tfrac{1}{2}\right) &= col\left(x_1\left(\tfrac{1}{2}\right), x_2\left(\tfrac{1}{2}\right)\right) = col\,(\lambda_1, \lambda_2). \end{aligned} \tag{160}$$

Using (160), the boundary restrictions (159) can be rewritten as

$$Ax(0) + x\left(\frac{1}{2}\right) = Az + \lambda - g\,(z, \eta, \lambda)\,, \tag{161}$$

where

$$\begin{aligned} z &= col\,(z_1, z_2), \\ \eta &= col\,(\eta_1, \eta_2), \\ \lambda &= col\,(\lambda_1, \lambda_2). \end{aligned} \tag{162}$$

Let us put

$$d(z,\eta,\lambda) := Az + \lambda - g(z,\eta,\lambda)\,, \tag{163}$$

where z, η and λ are given by (162).

Using (163), the parametrized boundary conditions (161) can be written in the form:

$$Ax(0) + x\left(\frac{1}{2}\right) = d(z,\eta,\lambda)\,. \tag{164}$$

It is easy to check that the matrix K from the Lipschitz condition (93) is

$$K = \begin{pmatrix} 0 & 0.05 \\ 0.5 & 1 \end{pmatrix},$$

and

$$r\left(K\right) < 1.03 < \frac{10}{3T},$$

when $T = \frac{1}{2}$.

Vector $\delta_D\left(f\right)$ can be chosen as

$$\delta_D\left(f\right) \leq \begin{pmatrix} 0.03125 \\ 0.515 \end{pmatrix}.$$

One can varify that, for the parametrized boundary–value problem in this example, all needed conditions are fulfilled. So, we can proceed with application of the numerical–analytic scheme described above and thus construct the sequence of approximate solutions.

The components of the iteration sequence for the boundary–value problem (155) under the linear parametrized two–point boundary conditions (164) have the form

$$\begin{aligned} x_{m,1}(t,z,\eta,\lambda) := z_1 + \int_0^t f_1\left(s, x_{m-1,1}(s,z,\eta,\lambda), x_{m-1,2}(s,z,\eta,\lambda)\right) ds - \\ -2t\int_0^{\frac{1}{2}} f_1\left(s, x_{m-1,1}(s,z,\eta,\lambda), x_{m-1,2}(s,z,\eta,\lambda)\right) ds + \\ +2t(z_2^2 + \eta_1 + 0.025 - z_1), \end{aligned} \tag{165}$$

$$x_{m,2}(t,z,\eta,\lambda) := z_2 + \int_0^t f_2\left(s, x_{m-1,1}(s,z,\eta,\lambda), x_{m-1,2}(s,z,\eta,\lambda)\right) ds -$$
$$-2t \int_0^{\frac{1}{2}} f_2\left(s, x_{m-1,1}(s,z,\eta,\lambda), x_{m-1,2}(s,z,\eta,\lambda)\right)) ds +$$
$$+2t(0.025 - z_1), \tag{166}$$

where $m = 1, 2, 3, \ldots,$

$$x_{0,1}(t,z,\eta,\lambda) = z_1 + 2t(z_2^2 + \eta_1 + 0.025 - z_1), \tag{167}$$

$$x_{0,2}(t,z,\eta,\lambda) = z_2 + 2t(0.025 - z_1). \tag{168}$$

The system of approximate determining equations depending on the number of iterations for the given example is

$$\Delta_{m,1}(z,\eta,\lambda) = -2\int_0^{\frac{1}{2}} f_1\left(s, x_{m-1,1}(s,z,\eta,\lambda), x_{m-1,2}(s,z,\eta,\lambda)\right) ds + \\ +2(z_2^2 + \eta_1 + 0.025 - z_1) = 0, \tag{169}$$

$$\Delta_{m,2}(z,\eta,\lambda) = -2\int_0^{\frac{1}{2}} f_2\left(s, x_{m-1,1}(s,z,\eta,\lambda), x_{m-1,2}(s,z,\eta,\lambda)\right) ds + \\ +2(0.025 - z_1) = 0, \tag{170}$$

$$x_{m,1}\left(\frac{1}{4}, z, \eta, \lambda\right) = \eta_1, \quad x_{m,2}\left(\frac{1}{4}, z, \eta, \lambda\right) = \eta_2, \tag{171}$$

$$x_{m,1}\left(\frac{1}{2}, z, \eta, \lambda\right) = \lambda_1, \quad x_{m,2}\left(\frac{1}{2}, z, \eta, \lambda\right) = \lambda_2. \tag{172}$$

Using (165)–(168) as a result of the first iteration $(m = 1)$, we get:

$$x_{11} = z_1 - 0.001666666667t^3 + 0.00125t^2 - 0.05t^2z_1 + \\ +0.04979166666t - 1.975tz_1 + 2tz_2^2 + 2t\eta_1,$$

$$x_{12} = z_2 + 0.002t^5 - 0.0008333333332t^3 + 0.06666666666t^3z_1 - \\ -1.333333333t^3z_1^2 + 0.5t^2z_2^2 + 0.5t^2\eta_1 + 0.875t^2 - 0.5t^2z_1 - \\ -0.05t^2z_2 + 2t^2z_2z_1 - 1.766666667tz_1 - 0.25tz_2^2 + \\ +0.006333333333t + 0.3333333334tz_1^2 - 0.25t\eta_1 + \\ +0.025z2t - tz_2z_1,$$

for all $t \in \left[0, \frac{1}{2}\right]$.

The system (169)–(172), as follows from the first iteration above, now has the form

$$\begin{gathered}\Delta_{1,1}(z,\eta,\lambda) := -0.0500263021 - 1.975937500 z_1 - \\ -0.002083333334 z_1^2 + 2.001041667 \eta_1 + 0.004166666666 z_2 z_1 - \\ -0.05010416666 z_2 + 2.001041667 z_2^2 = 0,\end{gathered} \tag{173}$$

$$\begin{gathered}\Delta_{1,2}(z,\eta,\lambda) := 0.1631250001 z_2 z_1^2 - 0.04177083332 z_2 \eta_1 - \\ -0.1471874999 z_2^2 z_1 - 0.002083333338 z_1^2 \eta_1 - 000.8333333351 z_1^3 z_2 + \\ +0. - -6250000002 z_1^2 z_2^2 + 0.0005208333338 \eta_1^2 + 0.01989583334 \eta_1 z_1 + \\ +0.001041666668 \eta_1 z_2^2 + 0.004166666667 z_2^3 z_1 - 0.04177083332 z_2^3 + \\ +0.002116402121 z_1^4 - 0.04257275137 z_1^3 + 0.0005208333338 z_2^4 + \\ +0.006258188451 + 0.004166666670 \eta_1 z_2 z_1 + 0.3158912369 z_1^2 - \\ -0.9658448663 z_2 z_1 - 2.263765001 z_1 + 0.01775263207 z_2 - \\ -0.2503388207 \eta_1 + 0.7538330543 z_2^2 = 0,\end{gathered} \tag{174}$$

$$0.503125z_1 + 0.01249999999 + 0.5z_2^2 + 0.5\eta_1 = \eta_1, \tag{175}$$

$$\begin{gathered} 1.003125z_2 + 0.007041015625 - 0.4718750001z_1 + 0.06250000002z_1^2 - \\ -0.03125z_2^2 - 0.03125\eta_1 - 0.125z_2z_1 = \eta_2. \end{gathered} \tag{176}$$

The computation shows that the approximate solutions of the approximate determining system (173)–(176) are

$$\begin{gathered} z_1 := z_{11} = -1.732102940 \cdot 10^{-8}, \\ z_2 := z_{12} = -0.000005209304726, \\ \eta_1 := \eta_{11} = 0.02499998258, \\ \eta_2 := \eta_{12} = 0.006254548758, \\ \lambda_1 := \lambda_{11} = 0.0499999826, \\ \lambda_2 := \lambda_{12} = 0.02499480802. \end{gathered}$$

The first approximation to the first and second components of solution is

$$x_{11} = -1.73210294 \cdot 10^{-8} - 0.001666666667t^3 + \\ + 0.001250000866t^2 + 0.09979166608t,$$

$$x_{12} = -0.000005209304726 + 0.002t^5 - 0.0008333344879t^3 + \\ + 0.1000002604t^2 + 0.00008323804922t.$$

The error of the first approximation is

$$\max_{t\in[0,\frac{1}{2}]} |x_1^*(t) - x_{11}(t)| \leq 1.0041 \cdot 10^{-5},$$

$$\max_{t\in[0,\frac{1}{2}]} |x_2^*(t) - x_{12}(t)| \leq 6,8 \cdot 10^{-6}.$$

The error of the second approximation is

$$\max_{t\in[0,\frac{1}{2}]} |x_1^*(t) - x_{21}(t)| \leq 4.19 \cdot 10^{-9},$$

$$\max_{t\in[0,\frac{1}{2}]} |x_2^*(t) - x_{22}(t)| \leq 2 \cdot 10^{-6}.$$

The error of the third approximation is

$$\max_{t\in[0,\frac{1}{2}]} |x_1^*(t) - x_{31}(t)| \leq 1.51 \cdot 10^{-9},$$

$$\max_{t\in[0,\frac{1}{2}]} |x_2^*(t) - x_{32}(t)| \leq -1.264 \cdot 10^{-9}.$$

Application of the integral BVP.

Let us show an application of the numerical–analytic scheme, described above, for the system of differential equations

$$\begin{cases} \frac{dx_1}{dt} = 0.05x_2 + x_1x_2 - 0.005t^2 - 0.01t^3 + 0.1 = f_1(t, x_1, x_2), \\ \frac{dx_2}{dt} = 0.5x_1 - x_2^2 + 0.01t^4 + 0.15t = f_2(t, x_1, x_2), \end{cases} \tag{177}$$

with two–point integral boundary conditions

$$Ax(0) + \int_0^{\frac{1}{2}} B(s)x(s)ds + Cx\left(\frac{1}{2}\right) = d, \tag{178}$$

where $t \in \left[0, \frac{1}{2}\right]$,

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},\ B(t) = \begin{pmatrix} 0 & t/2 \\ 1/2 & 1/4 \end{pmatrix},$$
$$C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ d = \begin{pmatrix} 13/256 \\ 7/960 \end{pmatrix}.$$

It is easy to check that the exact solution of the problem (177), (178) is

$$\begin{cases} x_1^* = 0.1t, \\ x_2^* = 0.1t^2. \end{cases}$$

Suppose that the boundary–value problem (177), (178) is considered in the domain

$$D = \{(x_1, x_2) : |x_1| \leq 0.42, |x_2| \leq 0.4\}.$$

Let us introduce the following parameters:

$$
\begin{gathered}
z := x(0) = col\left(x_1(0), x_2(0)\right) = col\left(z_1, z_2\right), \\
\lambda := \int_0^T B(s)x(s)ds = col\left(\lambda_1, \lambda_2\right) \\
\eta_2 := x_2\left(\tfrac{1}{2}\right)
\end{gathered} \tag{179}
$$

Using (179), the boundary restrictions (178) can be rewritten as linear ones that contain already non–singular matrix C_1

$$
Ax(0) + C_1 x\left(\frac{1}{2}\right) = d(\lambda, \eta), \tag{180}
$$

where $\eta = col(0, \eta_2)$, $C_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $d(\lambda, \eta) := d - \lambda + \eta$.

It is easy to check that the matrix K from the Lipschitz condition (128) can be taken as

$$K = \begin{pmatrix} 0 & 0.05 \\ 0.5 & 0.8 \end{pmatrix},$$

and

$$r(K) < 0.84 < \frac{10}{3T},$$

when $T = \frac{1}{2}$.
Vector $\delta_D(f)$ can be chosen as

$$\delta_D(f) \leq \begin{pmatrix} 0.18925 \\ 0.3278125 \end{pmatrix}.$$

Domain D_β is defined by inequalities:

$$2t(0.05078125000 - \lambda_1 - z_1) \leq 0.0473125,$$

$$2t(0.007291666667 - \lambda_2 + \eta_2 - 2z_2) \leq 0.081953125,$$

$\forall \lambda_1, \lambda_2 \in D_0$, $\eta_2 \in D$.
The domain D_0 is such that

$$D_0 = \{(\lambda_1, \lambda_2) : |\lambda_1| \leq 0.105, |\lambda_2| \leq 0.31\}.$$

One can verify that, for the parametrized boundary–value problem (177), (180), all needed conditions are fulfilled. So, we can proceed with application of the numerical–analytic scheme described above and thus construct the sequence of approximate solutions.

The components of the iteration sequence for the boundary–value problem (177) under the linear parametrized two–point boundary conditions (180) have the form

$$\begin{aligned} x_{m,1}(t,z,\lambda,\eta) := z_1 + \int_0^t f_1\left(s, x_{m-1,1}(s,z,\lambda,\eta), x_{m-1,2}(s,z,\lambda,\eta)\right) ds - \\ -2t\int_0^{\frac{1}{2}} f_1\left(s, x_{m-1,1}(s,z,\eta,\lambda), x_{m-1,2}(s,z,\eta,\lambda)\right) ds + \\ +2t(0.05078125 - \lambda_1 - z_1), \end{aligned} \tag{181}$$

$$x_{m,2}(t,z,\lambda,\eta) := z_2 + \int_0^t f_2\left(s, x_{m-1,1}(s,z,\lambda,\eta), x_{m-1,2}(s,z,\lambda,\eta)\right) ds -$$
$$-2t\int_0^{\frac{1}{2}} f_2\left(s, x_{m-1,1}(s,z,\lambda,\eta), x_{m-1,2}(s,z,\lambda,\eta)\right))ds +$$
$$+2t(0.007291666667 - \lambda_2 + \eta_2 - 2z_2), \tag{182}$$

where $m = 1, 2, 3, \ldots,$

$$x_{0,1}(t,z,\eta,\lambda) = z_1 + 2t(0.05078125 - \lambda_1 - z_1), \tag{183}$$
$$x_{0,2}(t,z,\eta,\lambda) = z_2 + 2t(0.007291666667 - \lambda_2 + \eta_2 - 2z_2). \tag{184}$$

The system of approximate determining equations

$$\Delta_m(z,\lambda,\eta) = col(\Delta_{m,1}(z,\lambda,\eta), \Delta_{m,2}(z,\lambda,\eta))$$

depending on the number of iterations for the given example is

$$\Delta_{m,1}(z,\lambda,\eta) = -2\int_0^{\frac{1}{2}} f_1\left(s, x_{m-1,1}(s,z,\lambda,\eta), x_{m-1,2}(s,z,\lambda,\eta)\right) ds + \\ +2(0.05078125 - \lambda_1 - z_1) = 0, \tag{185}$$

$$\Delta_{m,2}(z,\lambda,\eta) = -2\int_0^{\frac{1}{2}} f_2\left(s, x_{m-1,1}(s,z,\lambda,\eta), x_{m-1,2}(s,z,\lambda,\eta)\right) ds + \\ +2(0.007291666667 - \lambda_2 + \eta_2 - 2z_2) = 0, \tag{186}$$

$$\int_0^{\frac{1}{2}} B(s) x_m(s,z,\lambda,\eta) ds = \lambda, \tag{187}$$

$$x_{m,2}\left(\frac{1}{2}, z, \lambda, \eta\right) = \eta_2, \tag{188}$$

$m = 1, 2, 3, \ldots.$

The computation shows that the approximate solutions of the determining system (185)–(188) for $m = 1$ are

$$\begin{gathered} z_1 := z_{11} = -4.253290711 \cdot 10^{-7}, \\ z_2 := z_{12} = 7.295492706 \cdot 10^{-7}, \\ \lambda_1 := \lambda_{11} = 0.0007814848293, \\ \lambda_2 := \lambda_{12} = 0.007290937121, \\ \eta_2 := \eta_{12} = 0.0249993271. \end{gathered}$$

The first approximation to the first and second components of solution is

$$x_{11} = -0.0025t^4 + 0.09968792498t - 4.253290711 \cdot 10^{-7} + \\ + 0.001249955722t^2 - 8.714713042 \cdot 10^{-8} t^3,$$

$$x_{12} = 0.00008047566353t + 0.002t^5 + 7.295492706 \cdot 10^{-7} + \\ + 0.1000000588t^2 - 0.0008332398387t^3.$$

The error of the first approximation is

$$\max_{t\in\left[0,\frac{1}{2}\right]} \left|x_1^*(t) - x_{11}(t)\right| \leq 2.1 \cdot 10^{-5},$$
$$\max_{t\in\left[0,\frac{1}{2}\right]} |x_2^*(t) - x_{12}(t)| \leq 2.2 \cdot 10^{-6}.$$

The error of the second approximation is

$$\max_{t\in\left[0,\frac{1}{2}\right]} \left|x_1^*(t) - x_{21}(t)\right| \leq -4.03 \cdot 10^{-8},$$
$$\max_{t\in\left[0,\frac{1}{2}\right]} |x_2^*(t) - x_{22}(t)| \leq 1.2 \cdot 10^{-6}.$$

Continuing calculations one can get more approximate solutions of the original boundary–value problem with higher precision.

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THANK YOU FOR ATTENTION!