# Bifurcations of global invariant manifolds in an optically-injected laser

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*Summary*. We study a model of an optically injected laser from the point of view of its global two-dimensional invariant manifolds near a codimension-two non-central saddle-node homoclinic bifurcation. We compute the relevant global invariant manifolds and connecting orbits for representative parameter values to unravel how they organise the overall dynamics of phase space for the different parameter regimes.

## Problem setting

We consider a model for a laser with optical injection from [9] which is given as the following system of ordinary differential equations:

$$\begin{cases} \dot{E} &= k + \left(\frac{1}{2}(1+i\alpha)z - i\omega\right)E, \\ \dot{z} &= -2Gz - (1+2Bz)(|E|^2 - 1), \end{cases} \tag{1}$$

where $E = x + iy$ is the complex electric field and $z$ is the number of electron-hole pairs. We are interested in the global organisation of the three-dimensional $(x, y, z)$-phase space of (1) near a non-central saddle-node homoclinic bifurcation —denoted $\mathbf{SNH}$ throughout—. At such a codimension-two point, an orbit $\Gamma_0$ is homoclinic to a saddle-focus equilibrium $p$ which, simultaneously, undergoes a saddle-node bifurcation. In this case, unlike codimension-one saddle-node homoclinic phenomena (also known as a saddle-node on a cycle), the homoclinic orbit $\Gamma_0$ does not converges to the equilibrium along its centre direction but lies on the two-dimensional stable manifold $W^s(p)$ of $p$; see [4, 5, 6] for more details.

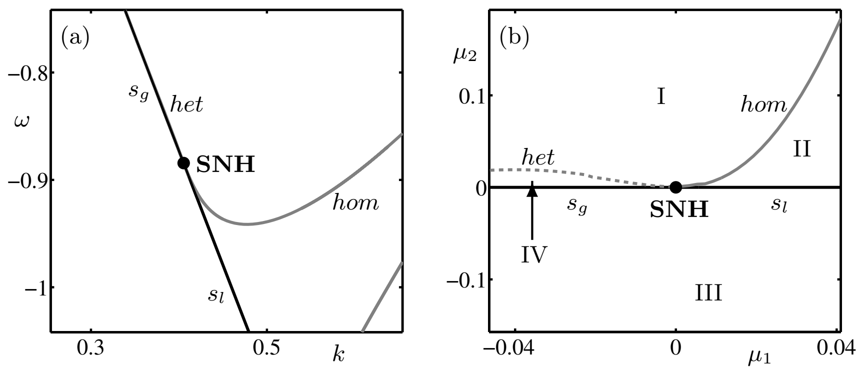


Figure 1: The bifurcation diagram near the $\mathbf{SNH}$ point $(k^*, \omega^*) \approx (0.40536, -0.88438)$ in (1) in the $(k, \omega)$-plane (a), and in the $(\mu_1, \mu_2)$-plane (b) after a suitable change of parameters. The other parameters are fixed at $\alpha = 2$, $B = 0.015$, $G = 0.035$.

Figure 1(a) shows the bifurcation diagram of (1) in the $(k, \omega)$-plane near a $\mathbf{SNH}$ point at $(k^*, \omega^*) \approx (0.40536, -0.88438)$. Figure 1(b) shows the same bifurcation diagram in some suitably rescaled parameters $(\mu_1, \mu_2)$. In both panels, a curve of saddle-node bifurcation is divided into two segments (denoted $s_g$ and $s_l$, respectively) by the $\mathbf{SNH}$ point. Along this bifurcation curve, the saddle-focus $p$ collides with a secondary (attracting) equilibrium $q$ in phase space. A locus of homoclinic bifurcation to $p$ —denoted $hom$— emerges from the codimension-two point. In addition, a curve of heteroclinic connection, denoted $het$, also converges to the point $\mathbf{SNH}$. Along $het$, a branch of the one-dimensional unstable manifold $W^u(p)$ forms a heteroclinic connection to $q$ lying on the two-dimensional strong stable manifold $W^{ss}(q)$ of $q$ [4, 5, 6]. While $het$ is very close to the segment $s_g$ in Figure 1(a), this is not an issue in Figure 1(b) where both curves are perfectly distinguishable from each other; furthermore, note that the locus of saddle-node bifurcation lies along the $\mu_1$-axis, and one can distinguish four open regions in the $(\mu_1, \mu_2)$-plane with different qualitative dynamics.

While the unfolding of homoclinic and heteroclinic orbits near a $\mathbf{SNH}$ point is well understood, it not so clear how these phenomena manifest themselves throughout the global phase space. Hence, our main objects of study are the global two-dimensional invariant manifolds $W^s(p)$ and $W^{ss}(q)$ of the equilibrium points $p$ and $q$, respectively, and their role as separatrices and organisers of the overall dynamics for the different parameter regions.

## Main results

We calculate the relevant two-dimensional manifolds as one-parameter families of solutions of a two-point boundary value problem which is implemented and solved in AUTO [7]. This technique allows us to compute and render the

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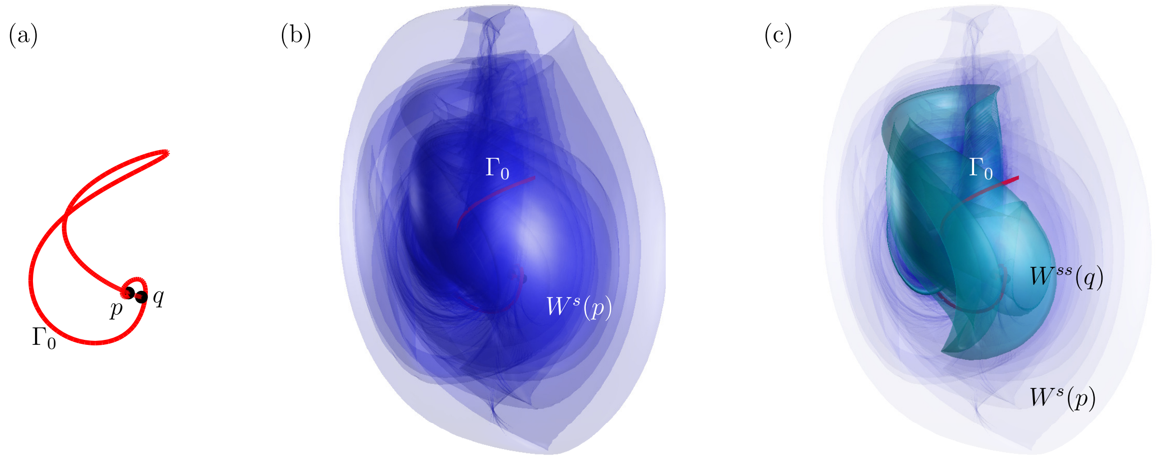


Figure 2: The relevant invariant manifolds at a homoclinic bifurcation in (1). Panel (a) shows the homoclinic orbit $\Gamma_0$ that connects to $p$ in both forward and backward time; panel (b) shows the two-dimensional stable manifold $W^s(p)$ as a (transparent) blue surface; finally, panel (c) shows how $W^{ss}(q)$ lies very close to $W^s(p)$.

two-dimensional manifolds as families of suitably chosen orbit segments; see [1, 2, 3, 8] for more details.
In the spirit of bifurcation theory, we obtain for each of the parameter regimes, a set of images of how the phase space is organized globally by the manifolds $W^s(p)$ and $W^{ss}(q)$, which are rendered, in a first step, as surfaces in $\mathbb{R}^3$; see Figure 2. We believe that the manifolds obtained with these techniques in this work represent the first illustrations of a global two-dimensional invariant manifold at the moment of codimension-one homoclinic bifurcation near a **SNH** point in a concrete system of differential equations. In particular, we are able to present the first images of how the global two-dimensional stable manifold $W^s(p)$ returns to the saddle-focus equilibrium at a codimension-two **SNH** point.
The unfoldings are presented by means of images of the intersection sets of $W^s(p)$ (which are curves) with a suitable sphere $S_R$ of radius $R$ centred at $p$. Since the sphere is a compact object, this approach enables us to analyse in a convenient way how the connecting orbits, the manifolds $W^s(p)$ and $W^{ss}(q)$, and the basins of attraction change as one varies the unfolding parameters $k$ and $\omega$. By doing so, we show the occurrence of a global saddle-node bifurcation of the two-dimensional manifolds $W^s(p)$ and $W^{ss}(q)$, which happens simultaneously to the local saddle-node bifurcation of the equilibria $p$ and $q$. Namely, as the point $(\mu_1, \mu_2)$ crosses a segment of saddle-node bifurcation, the global surfaces $W^s(p)$ and $W^{ss}(q)$ collide with one another and either disappear or move away, depending on the direction of the movement in the $(\mu_1, \mu_2)$-plane.

## Conclusions

The results of our investigation on model (1) provide a geometric explanation of how the two-dimensional stable manifold $W^s(p)$ of the equilibrium $p$ rearranges itself as a global object near a codimension-two **SNH** point, how it is created/destroyed at the saddle-node transition, and how the connecting orbits and the basins of attraction change in the process.

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